Alexander Ziw.

THE FIRST PRINCIPLES

OF

MECHANICS,

WITH

HISTORICAL AND PRACTICAL ILLUSTRATIONS.

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PREFACE.

Though several very instructive books have recently been published with the view of rendering the philosophy and application of mechanical principles easy and attractive to the general reader, I have not been able to find any work which seemed to me to supersede the necessity of such a one as I have here attempted. Its objects are to exhibit in a logical manner the grounds on which the doctrines of Mechanics rest;—to introduce the most important mathematical propositions which belong to the elementary portions of the science, without requiring from the reader any knowledge of mathematics beyond Geometry and the simplest processes of Arithmetic and Algebra;—and to shew that such a knowledge enables the student to understand and perform several of the calculations which the practical application of the subject involves.

The popular Treatises on Science which we possess are, so far as I have been able to judge, deficient in their bearing on one or other of these objects. It is not enough, if we would treat the subject philosophically, to state the leading mechanical facts which offer themselves, and to connect them with certain general laws and general terms loosely and hastily applied. The rules of sound reasoning ought to govern every exposition of science, however popular; and no wish to avoid wearying or perplexing the reader can set aside the obligation of the accurate use of terms and the paramount authority of logical connexion. It may not be needed, or wished, that

there should be a large apparatus of mathematical deduction, but it is quite requisite that our assertions should be as justly deduced from each other as if mathematics had been the instrument employed. It may not be necessary to solve recondite problems or to expound abstruse theories; but it is quite necessary that problems should not be referred to wrong principles, or difficulties made to seem easy by a lax use of language. However few and simple are the propositions introduced, it is requisite that each should appear in its proper sense, derivation, and bearing. This, even in the most elementary treatise, is essential to the value of the lesson taught; and is important in order that the doctrines of such an introductory and imperfect system may readily fall into their place, and harmonize with what is afterwards learnt, when the science is studied in its wider and more complete mathematical developement.

Mechanics, like every other physical science, involves two processes, the inductive and the deductive; the ascent from facts and observations to principles and axioms, to the most simple and general laws; and the descent again from such laws to their results in particular cases, their exemplification in special facts. The latter part of the science cannot be better presented than in the form of mathematical reasonings, such as usually constitute the main portion of treatises on mechanics: the modes in which the general laws have been obtained, and the grounds on which they rest, are more difficult to place clearly before the reader. This difficulty is increased by the manner in which some writers treat the subject; for they speak as if the ultimate axioms and most general principles of Mechanics were not only true, but self-evident independently of experiment; or at most, manifest by a reference to a few obvious facts. It appeared to me that the best mode of putting this matter in its true light was

to give a sketch of the history of each of the leading principles of the science. When it is seen through how many attempts, and after how many errors of the most intelligent speculators, every one of these doctrines has been reduced to its final simplicity and certainty, it will perhaps be more evident how entirely they depend upon experiment for their proof, and how far from easy the discovery of them was. In this way also the student will have his notice directed to some of the most natural mistakes which occur to a person whose object is to obtain true and distinct conceptions on these points.

In the deductive part of this Essay the mathematical reader will readily perceive that the propositions introduced are very few and limited compared with those which might have been given. My intention has been to avoid requiring from the reader, not only much previous mathematical reading, but also any great exertion of those processes of generalisation and abstraction which are seldom easy, except to persons of mathematical habits or mathematical minds. The propositions which I have given, few as they are, are capable of being applied to a vast number of numerical examples of all the most important elementary doctrines of Mechanics; and it is by such applications that I conceive the student may most readily obtain a clear insight into the principles of the science. The Chapter "on the Work done by Machines" will shew that these principles, without going higher in the theory than I have done, offer a wide field of practical utility.

It is, I hope, one of the privileges of an Elementary Essay like the present, to borrow freely, and without blame, from the best works which have been published. I have used this privilege to a considerable extent. In the illustrations of the fundamental Laws, Dr Arnott's deservedly popular 'Elements of Physics' contains well selected examples, stated with great

liveliness; and I have, in some instances, transcribed from such portions of his work, several sentences in succession. I have, in the same manner, borrowed an account of the progress of opinions concerning the second Law of Motion from Adam Smith's "History of Astronomy;" when the narrative occurs as a part of the history of the scientific tendencies of the human mind. In Sir John Herschel's admirable treatise "On the Study of Natural Philosophy," a statement is given expressing some of the most remarkable exertions of human labour in bushels of coals. It appeared to me that an Essay like the present was a place where the reader might reasonably expect to find an elementary explanation of the principle assumed in this estimate; namely, that any power may, so far as theoretical possibility is concerned, be employed to do any work; and that the work done will always be equivalent to the power expended. I have also borrowed an illustration of the accumulation of power from Professor Babbage's curious and valuable book "On the Economy of Manufactures." For calculations and information connected with the work done by atmospheric steam engines, I have availed myself of an instructive paper in the Philosophical Transactions, by Dr Davis Gilbert "*On the expediency of assigning specific names to all such functions of simple elements as represent definite physical properties.*" And for some other calculations concerning the steam engine, I have had recourse to Mr Tredgold's general treatise on the *Steam Engine.* I have introduced a brief exposition of the mode in which mechanical principles are to be applied to Locomotive Engines; having been in some degree induced to do this by the confusion which the problem appears to produce in the minds of many persons.

In writing of the history of the modern theory of Mechanics, it was impossible not to profit by Mr Drinkwater's "Life of Galileo." I gladly acknowledge great obligations to this ex-

cellent specimen of scientific biography. On one point however, I have ventured to express my dissent from the author of that work. I am not able to find, in the propositions concerning the equilibrium of weights on inclined planes, which he quotes from Tartalea's Edition of Jordanus, any good ground for deposing Stevin from the dignity of having been the first to give a proof of the statical property of the inclined plane.

Jordanus's proof confessedly assumes that it requires the same force to raise a body up any vertical height as to raise a body smaller in any proportion up a vertical height greater in the same proportion, *the bodies being supported on inclined planes.* Such a proposition, if asserted in 1300, or even in 1564, must have been, I conceive, a mere guess; since it was not obviously connected with any self-evident principle or known truth. It was probably one of many conjectures, and till better reason was shewn, had no claim to attention, above the solution of the problem of the inclined plane recorded by Pappus. To speak of the "principle of virtual velocities" as assumed in this solution, is attributing to the author a detection of analogies of which it is highly unlikely that he had any apprehension; and a generalisation which was not thought of till long afterwards.

Stevin's proof, on the other hand, does really refer the proposition to an axiom so clear as to compel conviction, though not the most simple which may be used. The impossibility of a loop of chain running perpetually over an inclined plane by its own weight, may be referred by *us*, if we chuse, to the "principle of virtual velocities;" but it was undoubtedly clear to the readers of Stevin on far less general views. The deducing the doctrine of oblique forces from this, as an axiom, was an important step in Mechanics. It adds to the merit of Stevin, that having been the person to make this step, he was fully aware of its use and importance. It applies in a very

great variety of cases of the properties of forces which he thus established; and in some of his works the inclined plane with the chain hanging round it, is employed as a vignette; accompanied with the motto "*Wonder en is gheen wonder.*"

I may add that Stevin's discovery of this proof is of an earlier date than I have stated in the following pages, if it be contained in the "*Beghinselen der Waaghconst*" published in 1586, which I believe it is, though never having seen the book, I cannot speak with certainty.

On these grounds I still consider Stevin as the first person who rightly solved the problem of forces acting obliquely, and consequently as the founder of the science of Statics. And I have no doubt that in this character he would have obtained far more celebrity than has fallen to his lot, if the speculations of Galileo had been given to the world half a century later than they were; and if, by this means, the science of Statics had been left to unfold itself upon its own proper principles, as in the reasonings of Stevin it had begun to do, and as it would have done if it had not become mixed with the mechanical doctrines of motion. But before the works of the Flemish engineer could produce much effect upon the mathematicians of Europe, the minds of physical philosophers were all turned towards Italy, where Galileo and his disciples were putting forth their doctrines concerning motion; a subject of much more varied and extensive bearings than the doctrine of equilibrium, and rendered peculiarly interesting by its connexion with the great question then agitated, of the truth or falsehood of the Copernican system. And the principle of virtual velocities, though in reality it was established as a general principle by being proved in each particular case, tended still further to throw into the shade these statical investigations, by making the doctrine of equilibrium appear to depend upon the doctrine of motion.

Having mentioned this subject, I will say a word on the fallacy thus introduced. It might have been considered that the principle of virtual velocities had not and could not have anything to do with the laws of motion. *Virtual* velocities, the comparative velocities which bodies may possibly have, in virtue of their connexion by means of a machine, are a purely geometrical conception; they depend on the properties of space alone, and are only a mode of expressing, by means of an hypothesis, the conditions to which the construction of the machine gives rise; they are not produced by forces, nor indeed produced at all, but only supposed. It is therefore certain that such merely notional and hypothetical "*virtual* velocities" have no connexion with physical and *actual* velocities produced by forces. To invest these geometrical phantoms with any of the attributes arising from physical laws, is a proceeding altogether arbitrary and illogical; nor can we, by such a supposition, any more than in any other way, establish *a priori* a connexion between the laws of rest and of motion. Nevertheless this principle, when it had once become familiar, was frequently so employed that the laws of equilibrium and of motion were involved in common assertions and arguments, instead of leaving the former to rest on strictly statical reasonings; and in various other ways also the purity and independence of Statics, as a mathematical science, were disregarded, by the mathematicians who were zealously following the rapid advance and rising fortunes of Dynamics. And though in other countries this mistake has been pretty well rectified, so that the two branches of Mechanics are now kept separate and distinct in all Treatises on the subject; with us the confusion still exists in some degree, and may be traced in several of the reasonings which are put forth, both in popular and in scientific publications.

ERRATA.

Page 51 line 13 *for* inclined plane Art. 48, *read* lever, Art. 29.

—— line 17 *after* before *insert* by the property of the Wedge, Art. 51.

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CHAPTER I.

SPECULATIONS WHICH LED TO THE ESTABLISHMENT OF THE FIRST LAW OF MOTION.

SECTION I.

Formation of the notions of Motion, Force, and Matter.

1. THE science of MECHANICS as the term is now understood, includes all the knowledge which we possess or can acquire concerning the motions of sensible portions of matter, considered in reference to the causes which occasion and modify the directions and velocities of those motions. The *Mechanical Philosophy* consists of the speculative inquiries in which we explain phenomena, or reason from them, with our views directed to their dependence upon the laws of motion and the properties of matter which bear upon those laws.

The science of Mechanics, like most other sciences, has gradually expanded to its present extent, having been at first confined within much narrower limits. Its name was conferred upon it while in its more restricted form; and has reference to the subjects of which it then principally treated; namely, to certain *Machines*, or material combinations employed for producing or preventing motion; the simplest of these machines, being those into which it was conceived that others might be resolved, were called the *Mechanical Powers*. These formed the first problems relating to this subject which were successfully treated by Mathematical writers. Archimedes proved the properties of one of these mechanical powers, called the *Lever*, and of a point called the *Centre of Gravity*, the properties of which depend upon those of the lever. The properties of other mechanical powers, the *Pully*, and the *Wheel and Axle*, may easily be shewn to depend on those of the lever. The properties of others of the mechanical powers, the *Inclined Plane*, the *Wedge* and the *Screw*, depend upon like principles, but

less obviously: and were not successfully investigated till the time of Galileo and Stevinus.

After that time the principles thus obtained, along with others, were rapidly extended to a great variety of problems, including many of the most prominent cases of the production or prevention of motion, as these take place among the objects which we see around us.

The consideration of such cases indeed, as problems attractive to the speculative powers of men, must have begun, as we know in fact that it did begin, at an early period of the exercise of man's intellect. The notice of *body* with its power of resistance, of *motion* with its essential conditions of space and time, of *force* as the cause producing and modifying motion, was inevitable when the faculty of abstraction was developed.

2. The conception of MOTION is suggested on all sides and at all times. Even in the earliest period of man's theoretical history, if we conceive the chariot, the ship, the bow, the sling, the potter's wheel, the axe of the woodman, the lever bar of the quarryman, to be as yet unknown, still motion would occur to the mind as one of the most universal affections of the objects which are perceived. The clouds sail slowly over the head of the contemplative observer; the rivulet glides swiftly at his feet, or leaps from rock to rock; the acorn or the apple drop from their native bough; or perhaps he bends down the struggling branch and plucks the detected fruit. He may see a stone urged slowly upwards, or rushing violently down, like that of Sisyphus.

ἤτοι ὁ μέν, σκηριπτόμενος χερσίν τε ποσίν τε,
λᾶαν ἄνω ὤθεσκε ποτὶ λόφον· ἀλλ' ὅτε μέλλοι
ἄκρον ὑπερβαλέειν, τότ' ἀποστρέψασκε κραταιΐς
αὖτις· ἔπειτα πέδονδε κυλίνδετο λᾶας ἀναιδής·

OD. XI. 595.

With hands and feet struggling he shoved the stone
Up to a hill top; but the steep well-nigh
Vanquished, by some great force repulsed, the mass
Rushed again, obstinate, down to the plain.

COWPER.

At a later period, every advance of the artisan supplies new examples of varied kinds of motion. The automaton, the clock, the watch, the windmill, the watermill, are successively brought into use. The thousand arms of labour, the myriad fingers of art and manufactures multiply the illustrations of the modes of motion, far beyond our reckoning: finally the steam engine adds its vast contribution to such instances; and the self-moving boat and the self-moving carriage exhibit their apparently spontaneous progression as the result of many motions which take place among the parts of their internal machinery.

3. Along with these countless forms of motion there is a frequent suggestion also of the conception of FORCE. Perhaps this conception arises originally with our own consciousness of the exertions by which we put objects in motion. We lift a stone, we hurl it from us; in all this we are conscious both of an impulse of the will and an effort of the limbs. We exert *force*, we *press*, we *thrust*, we *urge*, we *impel* the object.

λᾶαν ἀέιρας
ἧκ' ἐπιδινήσας, ἐπέρεισε δε ἶν' ἀπέλεθρον.

IL. VII. 268.

Then Ajax far a heavier stone upheaved,
He whirled it, and with might immeasurable
Dismissed the mass.

COWPER.

And such volition and such exertion, such force, pressure, impulse, in a great or less degree, with a consciousness more or less distinct, we are perpetually putting forth, when we move our own limbs, or change, or attempt to change the place, or to influence the movements of any inanimate object: when we stand or lean, walk or leap, strike or struggle, take up or lay down, cast or stop. This notion, of force exerted by ourselves, thus becomes one of the most familiar and frequent of those which occur to our minds, and is inseparably connected with our notions of any change which we can produce in the state of the bodies about us as regards rest or motion.

When we look at the various moving objects of which we have spoken, we easily perceive that their motions, when not influenced by our actions, are affected by various other causes and circumstances with which they are connected. The chariot is drawn by the exertions of its steeds, the bow propels the arrow by its springiness, the autumnal leaves are carried along by the current of the brook on which they are strewn, the heavier body in the balance descends by its weight and draws up the lighter to kick the beam. Though the cause may not be visible to the eye, we find no difficulty in admitting its operation; the ship is urged on by the breeze which inflates its sails, and even the distant clouds seem to obey the same impulse; and finally almost all bodies alike, the ponderous fragment torn from the lofty cliff, and the feather shaken from the wing of the sparrow, the unfrequent drop in the secret cave, and the innumerable flakes of the snow shower, appear to fall towards the earth by a universal and perpetual property of their nature, unrestricted by place, time or circumstance, unexhausted by size, number or repetition.

In considering the causes by which motions are thus produced or affected, if we fix our attention on the precise part of the body which is thus operated on by external causes, we conceive the action which takes place to be in a certain manner analogous to what takes place between our own limbs and the bodies on which we exert force or pressure in the manner which we have attempted to describe. The horses which draw the car, we cannot doubt, exert the same kind of action upon it, as we should have to exert in order to move it by our own strength. The bow, while we draw it, has resisted our pull by a force manifestly of the same kind as that which we apply to the string: hence, when we quit our hold, we may conceive that the string presses on the arrow by a force still of the same kind, though far more intense and urgent, than that which the hand could produce. The stream exerts a pressure which it requires an effort of human strength to resist, as he who fords a rapid river well knows: hence it may be conceived to urge forwards the bark in the same manner as the human hand might do. And even the invisible current of the blast is naturally imagined to produce its effect in the same manner as

the visible river. In this way all the influencing causes which affect the motions of bodies are assimilated to the agency which we ourselves exercise; are called *forces*, *pressures*; are said to *thrust*, *urge*, *impel*, *draw*, *pull*, the bodies on which they operate.

4. In the conception of the force which we thus exert, is invariably involved the notion of a *resisting* power residing in the matter on which we exert it. If we bend down the branch of a tree, the twigs which we take in the fingers prevent them from entirely meeting, while the elastic bough itself flies back, except we apply a sufficient strength. If we grasp a stone in the hand, if we shove it along the ground, our action is not effective in the first case till the closing of the hand is resisted by the solid texture of the stone, or, in the second, till we can surmount the opposition which the stone exerts, when it must rub against its resting place. Even if we scoop in our hand the fluid of the fountain, it resists in some degree the force by which we raise it to our lips; for if in its progress, the fingers open beneath it, the intended draught glides away in an instant. The whole of the agency which we immediately exercise upon lifeless objects is thus constantly connected with a feeling of their resistance when we operate upon them. In the poetical pictures of the shadowy regions inhabited by disembodied spirits, the earthly spectator sees what appear to him bodies capable of being acted on like those with which we are usually conversant;

> Et, ni docta comes tenues sine corpore vitas
> Admoneat volitare cavâ sub imagine formæ,
> Irruat ac frustra ferro diverberet umbras.

But when he makes the attempt, he finds that the images are impalpable like shadow or smoke; though conspicuous to the eye, they offer no resistance to the touch. And along with this removal of resistance, he loses also all power of influencing their motions by his bodily powers; the stone lies still on the ground when he has attempted to lift it; the beautiful form glides unfelt from the passionate grasp of the lover,

> ——neque illum
> *Prensantem nequicquam umbras* et multa volentem
> Dicere, preterea vidit.

This notion of resistance, combined with extension, appears to constitute our conception of MATTER. And this definite resistance exists not only when the matter is solid and unchangeable in its form by any force which we can immediately exert upon it, but where by its nature it can yield to the touch. The potter's clay resists the slight pressure with which the artist smooths its surface, though a greater force might change its shape. The softest couch, when it has yielded for a certain way to the incumbent weight, supports it in the place it has then reached: the stream opens a path to the arms of the swimmer, but not without resistance, and may even be so powerful as to overcome his efforts; and though when the air is tranquil the movements of our limbs are made without our being conscious that the element touches us on every side, the gale against which we are obliged to lean with all our might, makes us feel that even this thin fluid does not yield except to an adequate force. All matter, hard or soft, solid, fluid or aerial, whether or no it resumes its form when the constraint is removed, resists the constraint which external objects exercise upon it, and by this resistance receives and exercises that action by which the movements of bodies are influenced. *Motion* may be exemplified in impressions belonging to the eye only; but *force* and *matter* appear to be universally coexistent and correlative; and the conception of them includes feelings belonging to the touch, or rather, it would seem, to the animal *nisus*, by which we exert our muscles and move our limbs.

SECTION II.

Introductory attempts to obtain Laws of Motion.

5. WHEN we look at the various kinds of motion which offer themselves to our notice, and some of which we have pointed out, we observe in them various differences of quick and slow, of regular and irregular, of permanent and transitory. The intellect is soon led to make attempts to classify such phe-

nomena, and to ascend by classification and separation, to their causes and laws. Attempts of this kind were very early made with regard to the facts of motion, but at first these essays of the scientific faculty were far from happy or successful.

Among the different kinds of motions, a difference which at first seemed likely to be important and essential was their various apparent disposition to permanency. Most of the motions which external agents impress upon bodies cease after some time, and the body stops. We throw a stone, and after a short flight it falls to the ground and rests; a ball rolled along the floor runs but for a limited space; the ship soon lies motionless upon the waves when the winds cease to propel it; the schoolboy's top spins rapidly for a time, but finally flags and falls; the swing in which his companion tosses him, if left to itself, oscillates a while, but at last hangs at rest. These motions seem as if they decayed by their own nature. Other motions, on the other hand, go on with no such symptoms of decay. A stone is always ready to fall to the ground, and if dropped from the top of the highest cliff, its downward motion goes on faster, not slower, in consequence of the greater space it has to travel. The legend of Vulcan's fall from the skies, of nine days, shews how naturally we admit the permanency of this kind of motion. The river glides on steadily, and it is only a strangely rustic mind which expects its flow to cease.

> ——at ille
> Labitur et labetur per omne volubilis ævum.

The heavenly bodies offer still more impressive instances of undecaying and unvarying motions, and almost irresistibly suggest to the mind the endless and immutable course of eternity. "The stars that rise and fall;" the planets which perpetually pursue their "mystic dance," so complex yet so regular; the sun that daily "comes out of his chamber and rejoices to run his course;" "the inconstant moon" so constant to "the monthly changes of her circled orb;"—all supply instances of motions which seem not to share in the tendency to retardation, and semblance of weariness, which exist in similar motions when produced on earth.

An attempt was made to advance a step in our knowledge of motions by classifying them into *natural* and *unnatural;* a classification obvious perhaps, but quite untenable and unprofitable. The motion of a body by which it falls downwards was said to be *according to nature;* and this was asserted to be the reason why it continues undiminished, or even becomes more active as it proceeds. The motions of the celestial bodies were in like manner circular and permanent, because they are also according to nature. But the motion which we impress upon a body, by pushing or rolling or throwing it, is *contrary to nature;* for if we had left the body alone, it would by its nature have remained at rest. Hence, it was said, motions of the latter kind are not permanent; they speedily diminish and terminate: they have these characteristics of all that is violent and constrained. Thus it is that the projected stone, the whirling wheel, the top, the swing, gradually lose their motion when our agency ceases. In order to produce permanent motion we must exert permanent effort. The wheel of the artisan must be kept in play by his foot or hand; the carriage must be constantly pulled or it stops; the cradle is only to be kept rocking by continual small impulses. All matter seems perpetually ready to get rid of these unnatural motions, and to resume its congenial repose.

Such is the view of the law of motion entertained by Aristotle. It is further illustrated by his followers, who remark that motion—that is, such *unnatural* motions as we have been speaking of—are qualities like heat or cold; and as a heated body removed from the fire retains its heat but for a time, so it retains its motion also for a time only.

6. It is important for us to see that this doctrine is altogether erroneous. The motion in such cases, and in all cases, diminishes and finally ceases, not by a property belonging to the nature of motion, but by the agency of external bodies; by the influence of *retarding forces.*

The most important of these retarding forces are *friction* and the *resistance* of the surrounding air, or other fluid in which the motion takes place.

It is easy to perceive that the rubbing of a body in motion against another body, has the effect of diminishing the motion, and that this effect is greater or less according to the kind of the rubbing surfaces. If we drag a heavy mass along a floor of rough earth, we move it with difficulty: if the mass is placed on a smooth marble pavement, the same task becomes easier: if it rest on a sheet of ice or hard snow, a much smaller effort will urge it onwards. In the latter case indeed the motion is easily made to continue long with little diminution; the sledge of the Esquimaux glides rapidly over such a surface, with a slight exertion of the dogs which draw it. In like manner other motions continue longer when the moving surfaces and fixed surfaces which are in contact, are made smoother. The upper wheel of an overturned carriage may be made to spin freely for some time, if the axle be smeared with a proper preparation; a bowl resting with its smooth convex surface on a marble table, will rock for a long while. By thus diminishing the roughness or tenacity of the surfaces which undergo friction, we find that the motion is extinguished much more slowly, and goes on much longer, than would be the case without such precautions.

In a similar manner it is easy to perceive that the air or water in which a body moves offers a resistance to its motion, and tends to stop it altogether. A ship of which the prow is very blunt, sails sluggishly; if the extremity of the vessel be made of a form more suited for cleaving the water, her rate of sailing is increased. A cloak hung losely on a peg swings perhaps for a short time, but its regular oscillations soon cease; if it be twisted into a slender roll, it swings regularly for a much longer period. The diminution of motion in air or water thus becomes more rapid in consequence of an obtuse form, a broad surface; and the motion is more slowly destroyed when we acuminate the form or contract the surface.

Hence it appears evidently that the diminution and extinction of motion which was attributed to the motion itself, does in fact depend, at least in part, and in many cases, on external obstacles to the continuance of the motion, on friction and resistance. This follows clearly from the increased continuance of the motion when these obstacles are diminished. The slipperiness of the floor, the smoothness of the axle, the

sharpness of the prow, cannot add anything to the motion; they can only allow it to proceed with less check than it would otherwise have. However far we carry these facilities, we only give the motion an opportunity of shewing how it would proceed if it were entirely unimpeded. It is clear therefore that in common cases much of the retardation is owing to external causes; that the body, in a great measure at least, is retarded, not by the nature of motion, but by the action of force.

The question then arises whether *all* the retardation arises from the action of external forces: whether motion, altogether unimpeded, would be uniform and endless. We assert that it would be so, and this proportion is the basis of the science of Mechanics.

The following is the form in which the proposition is stated.

SECTION III.

The First Law of Motion.

7. *A body in motion will go on moving uniformly in a straight line, except so far as its motion is affected by the forces which act upon it.*

To prove this proposition positively and rigorously is not easy; for we cannot adduce any motion among terrestrial objects, which does of itself continue uniform; or which has gone on for a time exceeding known limits. But we can point out cases in which, the retarding forces being diminished, motions continue for a very long time. A heavy wheel on a smooth axle will spin many minutes: the friction may be diminished by *friction wheels*, and the motion continues longer still. For a particular purpose (namely, in order that its upper surface may be used as an artificial horizon for astronomical observations made on ship board) a top has been constructed so as to spin like a schoolboy's top; and by making its weight considerable, and its point of support small, smooth, and hard, the motion impressed upon it will continue for two or three hours. A heavy pendulum used in certain experiments has been known to go on swinging perceptibly for 10 or 12 hours, without

any new impulse. And in all these cases, if we place these kinds of apparatus in the receiver of an air pump, and by extracting the air, remove its resistance, the motion will continue for a very considerably longer period than in the common atmosphere.

Hence if there be any natural tendency to retardation, independent of external circumstances, it must be very small; since the retardation in these cases is so. But moreover in these cases, the external impediments to motion, though much diminished, are not entirely removed. There is still some friction, however hard and smooth be the surfaces which are in contact. We can rarify the air, but cannot produce an absolute vacuum; there is therefore even in the "exhausted receiver" some resistance. And the natural retardation of motion must therefore be considerably smaller than the smallest actual retardation which accurs in our experiments.

In this way the natural retardation of motion is shewn to be so small, that if it exist, it cannot be of much consequence in any common calculation. The general and leading fact is, either exactly or approximately, that *motion is naturally uniform*.

8. In reality however, this law is not approximately but exactly true. For if we suppose it to be true, and on this supposition deduce the rules of the effects of forces, and by these rules calculate the amount of the small retardation arising from unremoved obstacles, which existed in our most perfect experiments, we find this calculated amount to agree exactly with the whole observed amount of the retardation of the motion: thus leaving nothing to be accounted for by the natural tendency of motion to diminish of itself.

9. Since in all material contrivances, however perfect, there is some resistance to motion, no machine can go on working for an indefinite period without a constant supply of force. Such a supply is accordingly provided in all machines which are intended to go for a long time. In a clock and a watch we administer this supply when we wind them up. The pressure then exerted, small as it is, is by the machinery

spread over the period, a day or a week, during which they go. All automatons are in the same manner provided with force. In a water wheel the perpetual descent of the water ministers this force; and this supply may be perpetual, because the various powers of nature (evaporation, currents of air, &c.) carry the fluid back to the higher regions from which it descended. In the steam engine the elasticity of the vapour is called into play by the continued agency of fire and the continued consumption of fuel. In electric and chemical actions a durable movement seems capable of being produced only by processes which in some degree destroy the materials. And as the smallest perpetual subtraction constantly repeated, would exhaust the largest quantity, so the smallest perpetual resistance, uncompensated, would finally destroy any original motion. And as in all combinations of matter there is some resistance, the doctrine that motion if undisturbed is endless, leads immediately to the conclusion that a *motion* actually *perpetual* is practically impossible.

10. The motions of bodies on the earth would therefore, if it were not for their mutual interference and contact, go on for ever as the motions of the heavenly bodies do go on. And we may now ask whether the motions of the celestial bodies are really different in their nature and laws, from those of terrestrial objects; and whether they be not in fact governed by the same laws of motion; the difference being, that the natural tendency of motion to continue unabated, which is here always checked and counteracted, operates there effectively and predominantly, there being, in the spaces in which the heavenly bodies move, nothing to resist or impede their motions.

We shall find irresistible reasons to believe that this is the case, when we come to see how precisely all the motions of the heavenly bodies, in the minutest particulars, are accounted for by supposing them governed by the same laws of motion which obtain here upon earth.

11. As no motion is naturally retarded, so neither is any motion naturally, and without external agency, accelerated, although this was at first supposed to be the case with the

motion of falling bodies. A stone, as it falls from a considerable height towards the earth, certainly falls quicker and quicker: but this acceleration arises from the perpetual action of a force; namely its gravity, or the earth's attraction, which is the same thing. It is obvious that the perpetual action of pressure or force of any kind in the same direction will communicate an increasing velocity. Thus a man who turns a heavy wheel or grindstone can urge it to its greatest speed only by an effort continued for some time. And it will easily be seen that all bodies, whether at rest or in motion, are always acted upon by a downward force which gives them weight, and makes them fall if not supported. When a man descends a ladder, carrying a load, the load continues to press upon him as he descends, and however rapidly he descends, if he descend at a steady rate; thus proving that the same force which urges a body at rest to descend, also urges a descending body to descend still more rapidly. Hence it appears that the same agency which is the cause of weight, will also be a cause why a falling body must fall with a motion increasing as it continues. The force of gravity acts upon the body as it falls, perpetually pressing upon and urging it downwards during the whole of its descent. The greater is the height from which a body falls and the greater speed will it have acquired; a stone dropt from a high cliff rushes past a spectator at the foot of the cliff almost too rapidly to be seen. And we now see that this fact is consistent with the doctrine that motion is naturally uniform; and the deviation from uniformity, here as in the former cases, arises from external forces; accelerating forces here, as retarding forces there.

Hence we are led to the conclusion that there is no such distinction of motions of different kinds, as we have spoken of: that all motion is naturally uniform; and that the different degrees of transitoriness or permanency, of increased or diminished velocity, depend entirely on the agency of external causes.

12. In the same manner in which it was imagined that a body in motion had of itself a tendency to resume a state of rest, it was imagined also that a body at rest had of itself a certain definite tendency to remain at rest, and would not be

put in motion except this tendency were overcome. This also, like the former supposition, is erroneous: the only tendency to rest which exists in a body, arises from the external forces which it is necessary to overcome in order to put the body in motion. If it were not for these, the smallest force would put the largest body in motion, though the motion would be slight according to the smallness of the force and the largeness of the body. Friction, in this as in the former case, makes it seem as if a resistance to motion existed in the body itself. When a heavy body rests on the level ground, a slight push produces no motion whatever in it: and it is only when we exert a certain considerable pressure that we move the mass. It was supposed at first that the force thus requisite to move a weight on level ground depended upon the weight. Pappus has preserved to us a problem current among the mathematicians of his time, in which this is taken for granted. "Having given the force which can move a given weight along a horizontal plane, to find the force which can move the same weight along a given inclined plane." The enunciation, as well as the solution, of this problem is entirely erroneous. The force requisite to move a body along a horizontal plane depends upon the state of the surfaces, as well as the weight of the body. On a hard sledge on smooth ice the requisite force would be less than on smooth earth; on smooth earth less than on rough. It might be made very small by placing the body on wheels; and if the mass were suspended by a cord, it would be found that a very slight force would impress a perceptible horizontal motion upon it, the friction being here removed. If, in the last case, the slight force be applied repeatedly, at such intervals of time as the oscillations of the suspended body would occupy, the motion may become considerable, with a very small expenditure of force. A very heavy pendulum may in this way be put in motion by the breath.

13. Thus body has no natural preference either for a state of rest or a state of motion. It is entirely passive. If it be at rest it remains at rest; if it be in motion it goes on moving. In the former case the slightest force which solicits it to move, if not opposed by an equivalent force, is obeyed: in the latter

case the slightest obstacle which retards it, takes away something from its velocity; and if such action continue, will finally extinguish the whole motion.

The law that body is thus indifferent to rest and motion is called the *law of inertia.* This law is exemplified in instances where, in things composed of several parts, motion is suddenly produced or stopped in some of the parts: as when a chariot is overturned and the charioteer thrown over the horses heads; when a ship strikes against a rock and the passengers are dashed violently forwards. Here we have, during the motion, referred the persons to the carriage or vessel in which they are: but when the vehicle stops, we see that they are affected by the law of inertia, as independent bodies; they go on with the motion which they had, though the machine in which they were carried ceases to move. If, indeed, the chariot or the ship go moderately quicker or slower than before, the passengers share the new speed without any violent change; for the friction of the part on which they are supported, supplies a force which sufficiently augments or diminishes the velocity of their motion. A person sitting in a carriage is thrown back when the horses suddenly increase their pace; but having recovered this movement, he is thrown forwards, if they as suddenly diminish it. If we begin to push a vessel of water along a table, the water is at first thrown up behind, and when the motion ceases, it perhaps dashes over the brim in front. If we turn the vessel round its center horizontally, and observe the motion of the water by means of any mote which may rest on its surface, we shall see that the water at first does not turn with the vessel, and acquires a rotatory motion gradually only, as the outer parts successively drag the inner parts along with them; but when the rotatory motion has been acquired by the fluid, it still continues, though we stop that of the vessel.

14. The first law of motion asserts that a body left to itself, will move in a straight line as well as with a uniform velocity. This appears in all the cases already mentioned, for whenever the motion is curvilinear, this circumstance is the result of some constraint. Other exemplifications of the same

fact may be seen in the effects which are sometimes attributed to centrifugal force.

When a body revolves in any curve, it has a tendency to the outside or convex side of the curve, which is sometimes attributed to a centrifugal *force*, but which is in fact merely a consequence of the first law of motion: the body tends to *fly from the center*, only, because, by going in a straight line, it necessarily recedes from a fixed point. It flies *off*, only because it flies *in a tangent*. Hence the examples of centrifugal forces are illustrations of the first law of motion. Thus, the cord of a sling when a stone is whirled round in it, is kept tight by the effort of the stone to obey the law of inertia, this effort thus producing a tendency to recede from the center. A wet brush, twirled on its handle, throws off the water, each particle, when it separates from the brush, following the course which it was pursuing at the last moment of its adhesion. A tumbler of water, placed in a sling or hoop, may be made to vibrate or to turn so that sometimes it is inverted and yet the water does not fall out, the fluid being more strongly urged upwards by the first law of motion than it is urged downwards by gravity.

"Carriages are often overturned in quickly rounding corners. The inertia carries the body of the vehicle in the former direction, while the wheels are suddenly pulled round by the horses into a new one. A loaded stage-coach running south, and turned suddenly to the east or west, strews its passengers on the south side of the road."

"A man or a horse turning a corner at speed, leans much inwards or towards the corner, to counteract the centrifugal force, that would throw him away from it."

"In skaiting with great velocity, this leaning inwards at the turnings becomes very remarkable, and gives occasion to the fine variety of attitudes displayed by the expert; and if a skaiter, in running, finds his body incline to one side, and in danger of falling, he merely makes his skait describe a slight curve towards that side, and the body by the law of inertia, or centrifugal force, refusing as it were to follow in the curve, restores the perpendicularity."

"It is owing to the centrifugal force in any bending part of a stream of water, that when a bend has once commenced, it

increases, and is soon followed by others, until that complete serpentine winding is produced, which characterizes most rivers in their course across extended plains. The water being thrown by any cause to the left side, for instance, wears that into a curve or elbow, and acts constantly by its centrifugal force on the outside of the bend, until rock or higher land resist the gradual progress; from this limit being thrown back again, it wears a similar bend to the right, and after that another to the left, and so on."

15. Force is thus requisite to produce any increase or diminution of velocity, and it will produce these effects by degrees. The transition may be slow, or it may be very quick, but it will always be gradual. In the case of a blow it *appears* to be instantaneous; yet the pressure is of the same kind, however different in force or duration, whether we strike or push suddenly, or press steadily. There are gradations of intensity and duration by which we may pass insensibly from one of these cases to the other. All force is of the nature of pressure; and that which we call impact, a blow, a stroke, is only intense and brisk pressure, suddenly begun and terminated.

If we place on the end of the finger a coin lying flat on a piece of card, a fillip given to the edge of the card will make it fly away, while the superincumbent coin is left resting on the finger. The coin rests, by the law of inertia, except so far as it is moved by the force applied to it. But this force is only the friction of the card during the short moment which the impact occupies; and such a force is insufficient to move the coin beyond the end of the finger.

16. The pressures which may produce motion may also produce rest. When we sustain a weight in the hand; when we pause with the bow after we have drawn it; when two weights exactly poise in a balance; when a machine holds motionless a mass which it has raised from the ground;—we have pressures employed, not in producing, but in preventing motion.

The relations which must subsist among pressures, in order that they may produce this effect, may be calculated from

simple principles. They form the first division of Mechanics. In this case, the forces which act upon the body, are said to *balance*, or to *destroy* each other, or to be *in equilibrium:* and the branch of Mechanics, which treats of forces so circumstanced, is called *Statics*. The other division, which is termed *Dynamics*, treats of the laws of *motion* as distinguished from those of rest, and requires other principles, which the examination of those of Statics will prepare us to apprehend.

CHAPTER II.

SPECULATIONS CONCERNING FORCES WHICH PRODUCE EQUILIBRIUM ON A LEVER.

SECTION I.

Introductory attempts.

17. A LEVER is a rigid rod moveable about a point: the point is called the *fulcrum* or center of motion: the portions of the rod, on the two sides of the fulcrum, are called the *arms*.

The Lever is a *straight lever*, when the arms are in the same straight line; a *bent lever*, when they are not.

The two arms of the lever are supposed to be acted on by weights or forces, which tend to turn the lever opposite ways.

The beam of a balance, or of a steelyard, is a lever, the fulcrum being the pivot of the machine. Each half of a pair of scissors, or of a pair of pincers, or of a pair of nutcrackers is a lever, the hinge being the fulcrum. A poker used in stirring a fire, a crow bar employed in raising a heavy weight, is a lever, the edge on which each of these instruments rests being the fulcrum. An oar used in propelling a boat is likewise a lever.

18. The notice of the ancients was very early drawn to the circumstance that two weights of different magnitudes may balance each other by means of a lever. Aristotle proposes the question in his "Mechanical Problems." (δ) "Why, "says he," do small powers move great weights by means of the lever? when they have, besides, the weight of the lever itself to move. The power moves the weight the more easily in proportion as it is farther from the fulcrum. And the reason is, that the end which is farther from the center, describes a greater circle; so that the body which moves the other, will, by the same force, be transferred through a larger space."

And again, (β) in treating a similar question, he says, "The shorter end is moved *more against nature* than the longer." Here the reasoning is referred to the distinction of actions according to nature and against nature; a distinction which could never have led to any true or useful view of the subject.

But in taking up the writings of Archimedes, who wrote a little later, we find that the subject has already assumed a form altogether different, and is established upon clear and satisfactory principles; so clear and satisfactory indeed, that to this day scarcely any thing has been added to the evidence and simplicity of the proof of the lever which we find in this ancient writer.

19. The principles on which the proof of the properties of the lever rests, as given by Archimedes, will be stated hereafter. They depend entirely on the relations of weights in equilibrium, and at rest. But, the properties of this and of other mechanical powers have often, by other writers, been referred to a principle which supposes the weights to be in motion, namely, the *equality of momenta.*

The *Momentum* is defined to be "the weight of a body multiplied by its velocity;" and it is a proposition, true, and hereafter to be proved, that when two weights are in equilibrium on a lever, their momenta would be equal if they were put in motion.

But this property is not self-evident, and cannot rightly be made the foundation of our reasonings concerning the lever; though it was made the foundation of the mathematical doctrine of this and other machines by the early writers on Mechanics; and is still referred to by some writers, even of the present day, as an ultimate principle.

It is sometimes said by such writers that the smaller weight must be placed at the extremity of the longer arm, in order that the greater velocity of its motion may *compensate* for the greater weight at the other arm.

Or the *momentum* of a body is considered to mean its power of producing motion; and therefore it is asserted that bodies will balance on a lever when their *momenta are equal.*

Or it is asserted that a lever merely enables us to make up, by *working longer*, for the deficiency of immediate force: to concentrate or accumulate our force instead of dispersing and diffusing it.

These statements are altogether erroneous as a matter of *reasoning*.

We treat of the equilibrium of a lever which is absolutely *at rest*, and will continue so for ever, so long as the relation of the weights continues the same. Hence, no part has any velocity, and therefore the greater velocity of one part cannot compensate the greater force of another part.

It is not shewn, in such attempts, that velocity, if it did exist, could compensate for weight; and still less that it could do so in the asserted proportion.

It is not shewn that bodies will balance *directly* when their momenta are equal: and if this were shewn, it would not follow that they will balance *on a machine* when their momenta are equal.

If we were to define the weight of a body multiplied into the *square* of its velocity, to be its *impetus*, it would be no more self-evident that bodies will balance when their *momenta* are equal, than it is that they will balance when their *impetus* are equal; which latter proposition is not true.

It is not true that the smaller weight *works longer* than the larger, and therefore it cannot, in this way, *make up* for its defect of magnitude.

The axioms on which our proof of the properties of the lever will be founded, are on the other hand self-evident as statical principles, and the reasoning is independent altogether of the consideration of motion.

20. Archimedes being in possession of the true doctrine of the lever, established also several propositions concerning the *center of gravity*, and investigated the position of the center of gravity in bodies of different figures. *The center of gravity of any body is the point where the body may be supposed to be wholly collected*, so far as the effect of its weight is concerned. The body itself extends through a certain portion of space; and thus includes innumerable mathematical points; but one of

these points is to be taken as the place of the body, for all the purposes of mechanical reasoning; being the point in which, if all the matter of the body were conceived to be concentrated, it would produce exactly the same statical effect as the body in its actual state does produce.

It was assumed that such a point existed in every body; that every body had a center of gravity. This assumption was however not self-evident, and the reasonings of Archimedes were therefore so far imperfect. For it is not self-evident that the point at which different weights must be collected to produce *the same effect* as they did when dispersed, is the same point for different kinds of effects. Thus, in fig. 5, two equal weights P and Q must be collected at W, the middle point between them, in order that they may produce the same *direct pressure* as they did at P and Q; namely, a pressure equal to their sum. But is it manifest that when they are collected at the middle point W they will produce the same effect to *turn* the lever about the fulcrum A, as they did when at P and Q? This seems not to be equally clear; for we do not know, without some reasoning, how the effort of P, to turn the lever round A, will be modified and influenced by the action of Q exerted at the same time. Hence it is not self-evident that, for the purpose of keeping *this kind* of effect unaltered, the weights must be collected at the middle point between them.

The proof of the lever, given by Archimedes, makes it a *postulate* that equal weights acting at equal distances from a point in a lever will balance on that point; and it *assumes* that the effect which weights collected at this point exert, to turn the lever round any other point, is not altered by collecting them.

The postulate may properly be stated as an axiom; but the assumption was justly considered a blemish in the proof. Professor Vince, in the Philosophical Transactions for 1794, has given a proof of this assumption, deduced from axioms sufficiently self-evident. We shall state the proof of the property of the lever in the form thus given to it.

21. For the purpose of presenting this proof we must in the first place explain the manner in which forces are measured

when they are employed in producing equilibrium, that is in balancing other forces.

Since forces must be measured by their effects we might at first be tempted to say that the magnitude of the force is as the magnitude of the effects. But this rule would lead us into error, as will thus appear.

Let *MA*, fig. 2, be a straight spring fixed horizontally into a wall at *M*, and let a weight *P* hung to the end *A* bend it into the position *MB*; it would be erroneous to assert that a weight will be *twice as great* as *P* when it bends the spring into a position *twice as far* from the original position.

For this could not be generally true, except the proportion of the deflexion produced by the same weights were the same for springs of every possible size, strength, and elasticity, which is not obviously true at least, and is not actually true by experiment.

The true mode of obtaining a definition and a measure of the magnitude of forces is the following.

Section II.

On the Measure of Statical Forces.

22. We shall begin with the measure of *Weights*.

Any weight may be taken as the unit or primary standard of weight; for instance a common one pound weight of lead, (which is about $2\frac{1}{2}$ cubic inches).

Any other weight is *equal* to one pound, if it produce *the same mechanical effect* as the assumed unit.

Thus a certain mass (nearly $3\frac{1}{2}$ cubic inches) of iron, hung at *B*, (fig. 2.) will bend the spring *MA* through the same space as the assumed one pound of lead: therefore this mass of iron is also one pound.

In like manner we might obtain any number of masses, of any materials whatever, each equal to one pound.

Two of these masses together are a weight of two pounds, three are a weight of three pounds, and so on.

If C be now the point to which two of these masses, appended together, will draw the end of the spring, any other weight which, similarly appended, draws the end of the spring to C, will thus be known to be two pounds.

We may explain similarly the meaning of three, four, five, and any number of pounds; and these weights may be measured in a similar manner.

In the spring-balance this mode of measurement is actually employed. The places to which the spring is drawn by known weights are marked, and by means of these marks the magnitude of unknown weights is determined.

Instead of the bending of a spring, the same definitions might be established by the consideration of any other mechanical effect; for instance, all those weights are equal which will, when applied in the same balance and in the same manner, equipoise a given weight; and by the addition of such weights, any multiple of them is defined.

23. Any other forces (that is to say, *statical* forces, which we are here considering) may be measured in the same way as weights. For any other statical forces would produce exactly the same as those of weights; they would for instance, acting at the end of the spring MA, fig. 2, bend it through different angles according to the intensity of the force.

Any statical effect produced by force may be produced by a weight substituted for the force. Thus if a man pull downwards at a string, as at R, fig. 3, with a certain force, which is exactly counteracted by some resisting force, the effect of such pulling will be the same as if a certain weight P were suspended to the string; and this weight P is the measure or equivalent of the force at R.

Also if a force act obliquely, as in the direction of the string QO, fig. 4, and be such as exactly to counteract the force exerted at Q, the effect of the force in QO will be the same as if the string pass over a smooth nail or peg at O, and a certain weight P be suspended to its extremity. The weight P in this case is equivalent to the force in QO.

Statical forces may always be measured, as to magnitude, by the forces which will directly keep them in equilibrium; thus in fig. 3 and 4, the forces exerted by the hands pulling in opposite directions at Q and R, are necessarily equal.

The properties of the lever may be deduced from the following axioms.

24. Axiom I. *Equal forces acting perpendicularly at the extremities of equal arms of a lever to turn it opposite ways, will keep each other in equilibrium.*

For the forces act in a manner perfectly similar, and hence there can be no reason why one of them should prevail rather than the other.

We find that if one of the forces be greater, the arms remaining equal; or if one of them act at a longer arm, the forces being equal; the greater force or the longer arm preponderates, and the equilibrium is destroyed.

Cor. 1. Hence, the converse propositions are true; namely,

If two equal and opposite forces, acting perpendicularly at the extremities of a lever, keep it at rest, the arms are equal. For if they were not, the longer would preponderate.

If two forces, acting oppositely and perpendicularly at the extremities of equal arms of a lever, keep it at rest, the forces are equal. For if they were not, the greater would preponderate.

Cor. 2. If a weight, as W, fig. 5, be supported on a rod AB, on two fulcrums A and B, at equal distances from the weight, the pressures on the two fulcrums are equal.

Cor. 3. If two equal weights, P, Q, fig. 5, be supported on a rod PQ on two fulcrums A and B, situated so that PA, QB, are equal, the pressures on A and B are equal.

25. Axiom II. *If two equal weights balance each other upon a straight lever, the pressure upon the fulcrum is equal to the sum of the weights, whatever be the length of the lever.*

The whole weight is supported at the fulcrum of the lever, and hence it appears manifest, that the pressure which is there

supported must be equal to the whole weight, that is, to the sum of the two weights.

26. Axiom III. *If a weight be supported upon a lever which rests on two fulcrums, the whole pressure upon the fulcrums is equal to the weight.*

For here, as in the preceding Axiom, the whole weight is supported by the pressures which it exerts on these fulcrums, and therefore the sum of these pressures must be equal to the weight.

Thus if MN, fig. 7, be a lever moveable about C, on which two weights P, Q of 5 pounds and 2 pounds, balance each other, the fulcrum C supports a pressure of 7 pounds.

And if P, Q, fig. 5, be weights of 3 and 7 pounds respectively, the pressures on the two fulcrums A and B are together 10 pounds.

27. Prop. *If two equal weights act perpendicularly on a straight lever, they may be kept in equilibrium round any fulcrum by the same force as if they were collected at the middle point between them.*

Let P, Q, fig. 5, or 6, be the two weights, A the fulcrum, and W the middle point. Take $WB = WA$, and suppose a fulcrum placed at B.

When weights P and Q are supported on the lever, the pressure on each of the fulcrums is half the whole pressure, by Cor. 3 to Axiom 1; and the whole pressure is $P+Q$ by Axiom 3; therefore the pressure on each fulcrum is half $P+Q$.

When a weight W, equal to $P+Q$, is placed at the middle point, the pressure on each of the fulcrums is, by Cor. 2 to Axiom 1, equal to half the whole pressure; but the whole is $P+Q$, by Axiom 3; therefore the pressure on each fulcrum is half $P+Q$.

Hence, the pressure on the fulcrum B is in each case equal to half $P+Q$: and therefore the lever will in both cases be kept in equilibrium by the same force applied at B.

Cor. Hence, a horizontal prism or cylinder of uniform thickness and material, will produce the same effect as if it were collected at its middle point.

Thus, a cylinder *AD*, fig. 7, will produce the same effect on a lever *CA* as if it were collected at its middle point *E*: for this cylinder may be considered as composed of pairs of equal small weights (as 2 and 5) at equal distances from *E*, and each such pair will produce the same effect as if collected at *E*, and hence, the whole cylinder will produce the same effect as if it be collected there.

28. Prop. *Two weights acting perpendicularly upon a straight lever, on opposite sides of the fulcrum, will balance each other when they are reciprocally proportional to their distances from the fulcrum.*

Let the weights be any number of ounces, as for instances 3 and 5. Let them be divided into half ounces; there will be 6 of these in one weight, and 10 in the other; in all 16. Let these 16 half ounces be placed at equal distances along a straight lever *AB*, fig. 7, at the points 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16. Then it is evident, by Ax. 1, that the whole of the 16 weights will balance on the point *C*, mid-way between 8 and 9; for 8 and 9 will balance on this point, being equal weights and at equal distances; 7 and 10 will balance on *C*, for the same reason; as also 6 and 11, 5 and 12, 4 and 13, 3 and 14, 2 and 15, 1 and 16. Therefore the whole will balance on *C*.

But the weights 1, 2, 3, 4, 5, 6, will produce the same effect on the lever *AB* as if they were collected at *E*, the point mid-way between 3 and 4; for 3 and 4 may, by Article 27, be collected at *E* without altering the effect; as may 2 and 5, 1 and 6; and these 6 weights are 3 ounces. And in the same manner the weights 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 produce the same effect as if they were collected at *F*, mid-way between 11 and 12; for 11 and 12 may be collected at *E* (Art. 27,) as may 10 and 13, 9 and 14, 8 and 15, 7 and 16; and these 10 weights make up 5 ounces.

Hence if we have 3 ounces at *E* and 5 ounces at *F* the effect will be the same as before; that is, the weights will balance on *C*.

Now *CF* is evidently 3 of the equal divisions of *AB*, namely the two divisions from 9 to 10 and from 10 to 11, and

two halves of such divisions: and in like manner CE is 5 such divisions, namely 4 entire ones from 4 to 8, and two halves.

Therefore the weights 3 and 5 balance at distances which are reciprocally as 5 and 3.

And if, instead of the numbers 3 and 5, any others had been taken, the reasoning would have been the same and the result equally true:

Also if instead of an ounce we had taken any other unit of weight; and if the length of the lever AB had been different, the pressure upon the fulcrum C is still $5 + 3$ or 8; and is always the sum of the forces at E and F.

If instead of weights we take forces of any other kinds, the reasonings and the results will be the same.

29. Prop. *Two forces acting perpendicularly upon a straight lever, on the same side of the fulcrum and tending to turn it opposite ways, will balance each other when they are reciprocally proportional to their distances from the fulcrum.*

Let CM, fig. 8 or 9, be a lever as in the last proposition, whose fulcrum is N; thus let MN be 5 and CN be 3, and let the force at M be 3, and that at C, 5; then by the last proposition the two forces will balance, and the pressure on the fulcrum N will be $5 + 3$ or 8. Now let a force 8 be applied at N in a direction opposite to that in which the forces M and C act; this force will supply the place of the fulcrum, and

...st; and the lever is now kept at rest by

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...considered as a lever MC, moveable about

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...ve $MC = MN + NC = 5 + 3 = 8$, $NC = 3$,

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...n to be true in any other case.

...two weights or two forces, acting perpen-

...s of a straight lever, balance each other,

...their distances from the center of motion.

...shewn that when the weights are in this

...balance; and if the proportion be altered

by increasing or diminishing one of the weights, the effort to turn the lever round will be altered, or the equilibrium will be destroyed.

The Bent Lever.

31. PROP. *If two forces, acting perpendicularly on the arms of a bent lever, are inversely as the arms, they will balance each other.*

Let forces P, Q, fig. 12, act perpendicularly at M, N, on the arms of a lever MCN; and let them be such that

$$P : Q :: CN : CM;$$

they will balance each other.

Produce NC to D, so that $CD = CM$; and let the force P act at the arm CD instead of CM. The force will manifestly have the same tendency as before to turn the lever round C; for the only effect of the arm is to enable the force to turn the whole lever, and thus to counteract the force Q. But since $P : Q :: CN : CD$, the forces P, Q will balance each other on the straight lever DN. Therefore they will also balance on the bent lever MCN.

COR. 1. Conversely, if the forces act perpendicularly and balance each other, they are inversely as the arms.

COR. 2. If the arm CM or CN were bent so as to have any other form CFM, its extremity being the same, the same proportions would be true.

For CFM being perfectly rigid, if we join CM, the effect produced will be the same, if instead of the arm CFM we suppose the rigid surface $CFMC$ to be substituted. And in this surface if we remove any portion of the part towards F by a line parallel to CM, the effect will manifestly be the same as before. Hence, whether we have the rigid rod CFM, or CM, the effect will be the same.

32. PROP. *If two forces acting at any angles on the arms of any lever are inversely as the perpendiculars from the fulcrum upon their directions, they will balance each other.*

Let ACB, fig. 13, be the lever moveable about C; P, Q, two forces acting in the lines AP, BQ, and CM, CN perpendiculars on those lines. And let $P : Q :: CN : CM$; the forces will balance each other.

Let AM and CM be considered as rigid rods; then by Cor. 2, to last Art., the same effect will be produced whether we suppose the force P to act by means of the crooked arm CAM, or the straight one CM. In the same manner the force Q, acting at B, will produce the same effect as if it acted at N. But the forces P and Q, acting at M and N, would produce equilibrium, by last Article, because $P : Q :: CN : CM$. Hence, acting at A and B, they will produce equilibrium.

Cor. 1. $P \times CM = Q \times CN$.

Cor. 2. Conversely, if this proportion is true, or if these products are equal, the forces will balance.

When two forces balance each other on a lever or other machine, the one is considered as a force exerted, the other as a resistance to be overcome, or a weight to be supported: hence the two are distinguished as the *Power* and the *Weight*.

The equality of the momenta of the Power and Weight.

33. Prop. *If two forces, acting perpendicularly on the arms of any lever, balance each other, and if a small motion be given to the lever about its fulcrum, the velocities of the two points where the forces are applied, are inversely as the forces.*

Let the lever ACB, fig. 10, come into a new position aCb; then angle $aCb = ACB$, whence $ACa = BCb$, the sectors ACa and BCb are similar figures, and the arcs Aa, Bb, described by the points A and B, are therefore as CA and CB; that is, by Art. 31, reciprocally as P and Q. But the arcs Aa and Bb are as the velocities of A and B when the arcs are small. Hence the proposition is manifest.

Cor. Since A's velocity : B's velocity :: $Q : P$, we have $P \times A$'s velocity $= Q \times B$'s velocity.

It has already been stated that the momentum of any body is the weight of the body multiplied into its velocity (both

being of course expressed in numbers). Hence it appears, that when P and Q balance on a lever, their momenta are equal.

Examples of Levers.

34. If we know the force which, applied on any machine, will *support* a weight, or *neutralise* a resistance, any *greater* force will *raise* the weight or *overcome* the resistance. Hence, problems concerning weights to be raised or resistances to be overcome, belong to statics, so long as we have not to consider the time of producing the effect: and machines, the object of which is to produce motion, may be adduced as examples of levers.

Levers are sometimes divided into three kinds.

1st kind of Lever: the power and weight on opposite sides of the fulcrum.

2d kind of Lever: the power on the same side as the weight and further from the fulcrum.

3d kind of Lever: the power on the same side as the weight but nearer to the fulcrum.

The two first kinds give a *mechanical advantage* to the power, or enable a smaller force to balance a greater; the third kind is a *losing lever*, or requires a greater force to balance a smaller.

The losing lever is employed, not for gain of force, but for convenience of application.

The following are examples of Levers of the first kind:

The *handspike*, (a bar of wood of which one end is sharp and the other fitted for the hand to take hold of,) when the end is pressed down in order to raise a weight. A similar instrument made of iron, and having its extremity formed into claws, is called a *crow-bar*, and is a bent lever. The common *claw-hammer* for drawing nails is another example, the part of the hammer resting on the plank being the fulcrum.

Pincers or *forceps* are double levers of the first kind, the hinge being the common fulcrum. *Common scissors* are also

double levers of this kind; as are those stronger *shears*, with which bars and plates of iron are cut by the power of a steam engine, as easily as paper by the strength of the fingers.

The common *fire poker* is a lever, the bar on which it rests when used being the fulcrum, and the mass of coal raised the weight.

The common *scale beam* is a lever with equal arms; the *steelyard* is a lever with unequal arms; but, in order fully to understand this example, we must take into account the weight of the beam itself, which makes it necessary to refer this instrument to the case when more than two forces act upon a lever.

When a ship or boat rolls by the force of the wind, the masts move in the direction of the wind, but the keel moves the opposite way, and hence there is some point which for an instant moves neither one way nor the other, and may be considered as a fixed fulcrum for the time. With reference to this fulcrum, the mast, acted upon by the wind, may be considered as a long lever, lifting the lading and materials of the ship as weights.

A pair of pincers, used for drawing a nail out of a board by pressing on the board, is an instance of a bent lever, the part of the head of the pincers which rests on the board being the fulcrum, and the resistance of the nail being the weight. If the head of the pincers be convex all the way to the edge which holds the nail, and if the edges take hold of the nail close to the board, the fulcrum is, at the first instant of the action, close to the resistance, and therefore the perpendicular on the direction of the resistance is very small, and the resistance itself may be very great, and may yet be overcome.

The following are examples of the second kind of Lever.

A *handspike*, when the end is lifted up in order to raise a weight: a common *wheelbarrow*, the axis being the fulcrum: a *pole*, on which two parties carry a load, is a lever of this kind, each of them being the fulcrum with regard to the other; and in like manner, a bar (called in some places a *whipple tree*), by which two horses draw a plough. A *stock-knife* is a single lever, and a pair of *common nutcrackers*, a double lever of this kind.

When an *oar* is used, the handle in pulling goes forward, and the extreme point of the blade goes backward; hence, there is some intermediate point, which, for a moment, goes neither forward nor backward, and may be considered as a fixed fulcrum for the time. With reference to this fulcrum the oar is a lever of the second kind. But to estimate the effect upon the progress of the boat, we must take into account the reaction, which we shall hereafter treat of.

If two persons carry a weight on a stick, as MN, fig. 11, each end may be considered as a fulcrum with respect to the other. Hence, each person will support a portion of the weight, which portion is greater, in proportion as the distance of the weight from the supporting point is less. Thus, if CN be 4 times CM, $\frac{1}{5}$ of the weight will be supported at N and $\frac{4}{5}$ at M.

The following are examples of the third kind of Lever:—

The *hand* of a man *pushing at a gate* near the hinge, in order to open or shut it: the common *fire tongs*, the force of the fingers being applied near the hinge in order to grasp a coal with the ends of the instrument.

The limbs of animals are such levers for the most part. Thus, let EAB, fig. 14, represent the bones of the human arm, EA being the humerus, and AB the fore arm in a horizontal position; the latter is turned round the elbow A by the force of the muscular cordage DI, (which consists of two muscles, called the *biceps* and the *brachieus*). The mathematical point about which the fore arm turns, is the center of the convexity of the tubercle A; and from this point perpendiculars must be let fall on the forces which act, namely, the perpendicular OB on the vertical line in which the weight B acts, and OI on the direction DI in which the muscles act; and the force of the muscle will be as much greater than the weight supported, as OB is greater than OI.

According to Borelli, (*De Motu Animalium*, Prop. 22 & 23), when the humerus is horizontal the weight which a man can support in this way does not exceed 28 pounds; and in this case the perpendiculars just spoken of are as 20 to 1, the muscles acting close to the projection of the tubercle of the fore arm. Hence, the force of the muscles is 28×20 or 560 pounds.

When the humerus is upright, the weight which a man can support in this way is 35 pounds; and is greater than in the former case, because the muscle is now so situated that the perpendicular upon the direction of its action is greater than before. The muscle no longer lies close to the bone, but is retained near the hollow of the arm by certain membranous bands and by the external skin.

35. It has been seen that when two forces are in equilibrium on a lever we may compare them either by comparing their distances from the fulcrum, (Art. 28) or by comparing their velocities (Art. 32). The inverse proportion of these distances or of these velocities is the proportion of the power and weight.

If we take the former mode of estimation we refer to it by speaking of the *leverage* at which the power acts.

If we estimate the proportion of the forces by means of the velocities, the proposition is often stated thus, "what is gained in force is lost in velocity."

Thus, a boy who cannot exert a direct force of fifty pounds, may, by means of a claw hammer, draw a nail to which half a ton might have been suspended; because his hand moves through, perhaps eight inches, to make the nail rise one quarter of an inch. Hence, the velocity produced in the nail is less than the velocity of the hand in the ratio $\frac{1}{4} : 8$, and thus his force is increased in the ratio $\frac{1}{4} : 8$, or $1 : 32$, or $50 : 1600$.

Archimedes asserted that if he could find a fulcrum, he could, by his own strength, move the earth by means of a lever. But in producing this effect, his velocity must have been as much greater than that of the earth, as the weight he could exert would be less than the weight of the earth. Hence it may be calculated that he would have been required to travel with the velocity of a cannon ball, for hundreds of years, in order to alter the position of the earth by a small part of an inch.

The Equilibrium of several Forces on a Lever.

36. When more than two forces act upon a lever, the equilibrium requires a proportion among them which is easily collected from the preceding propositions. In the case of two forces, the forces and their distances being expressed by numbers, the products of each force into its perpendicular from the center of motion, are equal; *in the case of more than two forces, the products of the forces into their respective perpendiculars being taken, the sums of those in which the force tends to turn the lever one way and the other, must be equal.*

Thus, let four forces P, Q, P', Q' act on a lever MN', fig. 15, which is moveable about C; and suppose

$$CM = 8,\ CN = 5,\ CM' = 3,\ CN' = 6$$

$$P = 2,\quad Q = 1,\quad P' = 1,\quad Q' = 3$$

there will be an equilibrium; for

$$8 \times 2 + 5 \times 1 = 3 \times 1 + 3 \times 6;$$

but, if any one of the weights or distances be different from those expressed by these numbers, the other remaining as we have supposed them, there will not be an equilibrium.

In the preceding Articles we have left out of consideration the weight of the lever itself: we may consider the weight of the lever as collected in a single point, and apply the rule just given.

The steelyard is an example of a lever in which the weight of the beam requires to be considered.

The Steelyard.

37. The steelyard is a bar BK, fig. 16, suspended on a fulcrum C, and furnished with a weight P moveable along the longer arm CK. Bodies, of which the weights are to be determined, as Q, are suspended at the shorter arm at B, and their weight is known by the place to which P must be moved

in order to balance Q; the arm CK being *graduated*, that is, divided, and the divisions numbered according to the weight which we have to determine.

If the beam itself had no weight, the weight which P would balance would be proportional to its distance CM from C: and we should have $Q \times CB = P \times CM$. Hence, if E were the point where P must be hung to balance one pound at Q, F the point where it would balance two pounds, and so on, CE, EF, &c. would all be equal.

But the longer arm CK being the heavier, this mode of dividing (or *graduating*) the arm would not be correct.

Let D be the point where P must be hung so as exactly to balance the longer arm, Q being removed. Therefore, the longer arm balances a weight P at a distance at CD; and therefore, if we take CO equal to CD, the weight of the longer arm produces the same effect as the weight P acting at the distance CO would produce.

Therefore, when Q at B balances P at M upon the steelyard, Q balances a weight equal to P at the distance CO, besides the weight P at the distance CM; therefore, by what has been said in the last article,

$$Q \times CB = P \times CO + P \times CM;$$

or, since $CD = CO$, $Q \times CB = P \times CD + P \times CM$;

$$= P \times (CD + CM);$$

therefore $Q \times CB = P \times DM$.

Hence, when

$$DM = CB,\quad DM = 2CB,\quad DM = 3CB, \text{ \&c.}$$

we have $Q = P$, $Q = 2P$, $Q = 3P$, &c. respectively.

Therefore, the *graduation* of the arm must begin from the point D, and the divisions corresponding to each additional weight P, must be equal to CB.

These divisions may be subdivided in any proportion, and the points of subdivision will correspond to fractions of P in the same proportion.

The Wheel and Axle.

38. The Wheel and Axle is a machine consisting of a cylinder or axle AB, fig. 18, and a concentric circle or wheel EF, joined together, so that the whole is moveable about the axis of the cylinder: the weight W is attached to a cord NW, and will manifestly be raised or lowered as the wheel and axle are turned one way or the other. This weight is supported by a force applied at the circumference of the wheel EF; that is, either by another weight P acting by means of a string wrapped the contrary way to that at N, or by some other force as P', acting at a point M' in the circumference of the wheel.

PROP. *In the Wheel and Axle the power is to the weight as the radius of the axle is to the radius of the wheel.*

Let fig. 19 represent the machine seen endways with respect to the axle. The forces which tend to turn the machine opposite ways will still produce the same effect as before, if the axle be supposed to slide through the wheel till both the forces are opposite the same point C of the axis. But, in this case, the machine becomes a lever MCN, and the equilibrium will subsist, (by the property of the lever) if $P \times CM = Q \times CN$.

It is obvious, that in the state of equilibrium this is the same machine with the lever. When they are put in motion, the two machines differ. In the wheel and axle, the weight W ascends or descends in a vertical line: in the lever it describes a circular arc.

COR. 1. The power may act by means of a bar CM', and the wheel may be removed; this is the case in the *capstan* and the *windlass*.

COR. 2. If the direction of the power be not perpendicular to CM, we must draw a perpendicular upon it from C, and the proportion will be

$$P : W :: \text{rad. of axle} : \text{perp. on dir}^{n}\text{. of power.}$$

Pullies.

39. A small wheel moveable about an axis, and constructed so that a cord passes along a part of its circumference, is a *Pully*. The two parts of the cord on the two sides of the pully being in different directions, the pully serves to alter the direction in which the cord exerts its force.

A *fixed Pully* is one of which the axis is fixed; and such a pully serves only to change the direction of a force acting by means of a cord as O, fig. 4. It neither increases nor diminishes the amount of the force, the tension of the cord being the same throughout.

A single moveable Pully, as BA, fig. 20, in which the strings BC, AP are parallel, doubles the force P which acts at the string. For the tension of the string $PABC$ is the same throughout, and is equal to the force P; and hence the two parts AP, BC exert pressures each equal to P; and these together produce a pressure $2P$ on the axis of the pully at O. And therefore $2P$ is the weight which may thus be supported.

40. In the same manner the effect of any combination of pullies, with parallel strings, may be determined.

Thus, in fig. 21, the fixed pully B merely alters the direction of the string, therefore the tension of the string BA_3 is P, and the pressure upwards on the center of the pully A_3 is $2P$. Hence, the tension of the string $A_3\,A_2$ is $2P$, and the pressure upwards on the pully A_2 is $4P$; and the tension of the string $A_2\,A_1$ is hence $4P$, and the pressure upwards on the pully A_1 is $8P$, which is the weight supported.

In fig. 22 or 23, the same string goes round all the pullies, and its tension is therefore everywhere equal to P; and since there are four strings acting upwards at the lower block, the weight supported will be $4P$.

In fig. 24, the weight P produces a pressure upwards at C_1 equal to P, and a pressure downwards on A_1 equal to $2P$; this latter pressure produces a pressure upwards at C_2 equal to $2P$, and a pressure downwards at A_2 equal to $4P$; this latter

pressure, acting over the fixed pully A_3, produces a pressure upwards at C_3 equal to $4P$.

The whole weight supported will be the sum of the weights supported at C_1, C_2, C_3; that is, it will be $P+2P+4P$, or $7P$.

The Center of Gravity.

41. It has already been said that Archimedes assumed the existence of a center of gravity in a body, of whatever figure it might be; that is, he assumed that there was a certain point in which the whole matter of the body being collected, would always produce the same effect as the body itself produces.

Though it is, as we have already said, not satisfactory to start from this assertion as a postulate, the proposition is true, and may be proved by means of the doctrine of the lever.

The proof consists of two steps; in the first we define the center of gravity to be "the point about which the body or collection of bodies would balance itself in all directions;" and we shew how, from this definition, the center of gravity of any body or collection of bodies may be found; and in the second place we prove that if all the matter in the body or bodies be collected in this point it will produce the same effect as before; thus establishing what was improperly assumed.

We shall not here go through these propositions generally: we will only prove them in the case of bodies which lie in a straight line.

PROP. *To find the center of gravity of any number of bodies* P, Q, R, S *in the same straight line* (fig. 17).

Let the line PS be horizontal, and let the distances of Q, R, S from P be q, r, s respectively. Let G be the center of gravity, and let PG be x. Then $GP=x$, $GQ=x-q$, $GR=r-x$, $GS=s-x$. And by article 36,

$$P \times PG + Q \times QG = R \times RG + S \times SG;$$

$$\text{or } P.x + Q.(x-q) = R.(r-x) + S.(s-x),$$

$$\text{or } Px + Qx - Qq = Rr - Rx + Ss - Sx.$$

Transposing, $Px + Qx + Rx + Sx = Qq + Rr + Ss$

$$(P + Q + R + S)x = Qq + Rr + Ss$$

$$PG = x = \frac{Qq + Rr + Ss}{P + Q + R + S},$$

whence the place of G is known.

For example; let PQ, QR, RS be equal, and let P, Q, R, S, be 3, 5, 7, 9. Then $r = 2q$, $s = 3q$,

$$x = \frac{5 \times q + 7 \times 2q + 9 \times 3q}{3 + 5 + 7 + 9} = \frac{46}{24} q = 1\tfrac{11}{12} \times q,$$

the point G is at $\frac{11}{12}$ of the distance from Q to R.

It may be shewn that if P, Q, R, S balance upon G when the line PS is horizontal, they will balance upon G when the system is in any other position; hence G is the center of gravity by the definition just given.

42. Prop. *If any number of bodies acting on a horizontal straight lever be collected at their center of gravity, their effect to turn the lever about any fulcrum situated in that line will be the same as before.*

Let P, Q, R, S, fig. 17, tend to turn the lever about the point A. Then they will produce such an effect that the product of the force which acts to turn the lever the other way, multiplied into the perpendicular let fall from A upon the direction of the force, is equal to the sum of the products

$$P \times AP + Q \times AQ + R \times AR + S \times AS.$$

Let AP be called a; and let now the weights P, Q, R, S be attached to the lever at G; then the product of the force multiplied into the distance is

$$(P + Q + R + S)\,AG = (P + Q + R + S)\,(AP + PG)$$

$$= (P + Q + R + S)\,AP + (P + Q + R + S)\,PG;$$

or, putting a for AP, and for PG or x the value found in the last Article, the product is

$$= (P + Q + R + S)\,a + Qq + Rr + Ss$$

$$= Pa + Q(a + q) + R\,(a + r) + S(a + s)$$

$$= P \times AP + Q \times AQ + R \times AR + S \times AS$$

which is equal to the former product, and therefore the proposition is true.

In the same manner it is true in any other case, that the effect of any body or bodies, to produce equilibrium on any machine, is the same when they are collected at their center of gravity, as it was before they were collected.

CHAPTER III.

SPECULATIONS CONCERNING FORCES ACTING OBLIQUELY AT A POINT.

SECTION I.

Introductory Attempts.

43. THOUGH the Lever had been so early and so successfully treated, the next problem of Statics, the Inclined Plane, for a long course of centuries resisted the attempts of Mathematicians to solve it. In fact the confusion which prevailed in men's minds in consequence of the first law of motion not being yet clearly established, prevented them from stating the problem of the inclined plane to themselves in a proper form. Thus, as we have already said, Pappus enunciates the question thus; "Given a weight, and the power which will draw it along a horizontal plane, to find the power which will draw it on a given inclined plane."

But, independently of this confusion in the enunciation of the problem, the principles by means of which Pappus attempts the solution are altogether fallacious. He supposes the weight to be formed into a sphere and placed on the inclined plane, and he considers the weight of this sphere as supported by a lever, the fulcrum being the point of contact of the sphere with the plane, and the power being applied at the extremity of the horizontal radius. No reasonable ground is or can be assigned for identifying the effects of such a lever with those of the inclined plane for which it is thus substituted.

At a considerably later period Cardan seems to have conceived the problem more distinctly, though his endeavour to solve it is a mere conjecture. He saw that a body resting

on a plane requires no force to support it when the plane is horizontal, and requires a force equal to the whole weight of the body when the plane is vertical. He assumed that the force increased uniformly from one of these magnitudes to the other, and therefore that it is, for every inclination, as the angle of inclination.

Another mechanical writer who wrote a little later, Guidubaldi e Marchionibus Montis, (Marchmont) also attempted the problem of the oblique action of forces, with imperfect success. He considers the effect of the wedge, and comparing the direction in which it tends to produce motion with the direction in which the body thus acted on really moves, he observes that there is between these directions "a certain repugnance" which is greater as the angle of the wedge is more obtuse. He hence infers that the wedge will produce its effect the more easily the more acute it is, but without obtaining the exact proportion of the force. He also observes, rightly, that the screw may be considered as a wedge wrapped round a cylinder (*Mechanicorum Liber*. Pisauri, 1577.)

The person who first solved the problem of oblique forces, on principles which subsequent reasonings have confirmed, appears to have been Simon Stevin of Bruges, whose works were published soon after 1600. This mathematician not only deduced correctly the proportion of the power to the weight on the inclined plane, but, by means of the propositions which he thus established, resolved forces so as to obtain their effect in different directions, and solved a great number of the most important problems relating to the oblique action of forces. We shall explain briefly his mode of treating the subject.

It has been recently stated (*Drinkwater's Life of Galileo*, p. 82.) that the problem of the inclined plane had been solved at an earlier period by Jordanus in the 13th century, and that the work in which this solution was given, was published by Tartalea in 1565. As however this solution, even if it be interpreted so as to be right in the result, was mixed up with many of the usual Aristotelian errors on such subjects, and was not connected, so far as we know, either by the author, the editor, or the readers of the work, with any consistent and tenable train of mechanical reasoning, we may still, it

would seem, consider Stevin to be the father of Modern Statics, as we shall find Galileo to be the father of Dynamics.

After the Inclined Plane had been rightly reasoned upon by Stevin, various other authors also gave the solution of the same problem; and in a short time all questions connected with it were finally reduced to the general proposition of the resolution of forces.

We proceed to explain the reasonings of Stevin.

Stevin's Proof of the Force on the Inclined Plane.

44. Prop. *A weight resting on a perfectly smooth inclined plane, and supported by a string parallel to the plane, will be in equilibrium when the power is to the weight as the height of the plane is to its length.*

Fig. 26. An inclined plane is a plane inclined to the horizon, as *AC*; and its height *CB* is limited by a horizontal line *AB*.

Let there be a uniform chain or cord returning into itself, as *ACBD*, and let this pass round the plane *ACB*, and hang down below in the festoon *ADB*. This chain will remain at rest by its own weight. In the position of rest, the two sides of the festoon *ADB*, (the ends being in the same horizontal line) will be exactly similar, and will exert equal tensions at *A* and *B*. Hence if the part *ADB* be removed, the remaining part *ACB* will still continue at rest. But the weights of the portions of the chain *AC* and *BC* are as *AC* and *BC*. Hence the weight which rests on the inclined plane is to the weight which supports it as *AC* to *BC*.

Thus if the angle *CAB* be one third of a right angle (30^0) *BC* is the half of *AC*, and a force acting parallel to such a plane will sustain a weight double of itself resting on the plane.

In the same manner if we have two inclined planes of which the height is common to the two as *CA*, *CA'*, fig. 27, it may be shewn that the weights which rest upon them, and balance each other by means of strings parallel to the planes, are as the lengths of the planes.

Though the above reasoning proceeds on principles which are true, and which would probably be allowed to be sufficiently self-evident by most persons who have thought steadily on the subject, it will perhaps not be considered as the simplest axiom possible, that a loop of chain or cord, hung round an inclined plane, will remain at rest. Hence it is desirable to reduce the problem of the Inclined Plane, and similar ones, to still simpler principles; and it appears that they may all be made to depend on the Resolution of Forces, and that the Resolution of Forces may be deduced from the property of the Lever.

This is done in the following section.

SECTION II.

The Composition and Resolution of Forces.

45. PROP. *A force represented in magnitude and direction by the diagonal of a parallelogram is equivalent to two forces represented in magnitude and direction by the two sides, and acting at the same point.*

Fig. 28. Let a weight W be supported by two strings WR, WP. Let WS, a vertical line, represent the weight, and complete the parallelogram $WRSP$. Draw perpendiculars RE, RF on WS, WP.

In the condition of equilibrium we may consider the string RW as a lever, on which the weight W is supported by a force acting in the line WP. Hence, by the property of the lever,

$$\text{force in } WP : \text{force in } WS :: RE : RF.$$

But the triangles RSW, RPW are equal, being on the same base RW and between the same parallels RW, PS: and therefore their doubles are equal, that is the rectangles of WS, RE and of WP, RF are equal. Therefore

$$RE : RF :: WP : WS; \text{ hence}$$

$$\text{force in } WP : \text{weight in } SW :: WP : WS.$$

In like manner we should find

$$\text{force in } WR : \text{weight in } SW :: WR : SW.$$

Hence it appears that the two forces in WP, WR, which support the weight SW, are as the sides of the parallelogram WP, WR. And a force WS would support the weight SW; therefore the forces which are as WP, WR are equivalent to the force represented by the diagonal WS.

This proposition is called the parallelogram of forces.

Cor. 1. Since RS is equal and parallel to WP, RS may represent the same force which WP represents. Therefore the forces represented by WR, RS are equivalent to the force represented by WS.

Cor. 2. Any three forces represented in magnitude and direction by the three sides of a triangle, taken in order, as WR, RS, SW, will keep a point at rest.

Let a boat B, fig. 1, be kept at rest, being drawn by three persons in the directions BP, BQ, BR. In PB take any point S, and draw SR parallel to QB; the three forces which act in the directions BP, BQ, BR, will be proportional to the three lines SB, BR, RS.

46. This proposition is of very frequent use in mechanical reasonings. When a force acts so that it cannot produce its effect directly, it may often be resolved into two component forces, of which one is somehow counteracted or destroyed, while the other produces its effect.

Thus the wind blows against a kite, which is in a position inclined to the horizon; the motion of the air is horizontal, but the horizontal force being resolved into two, one along the surface of the kite and the other perpendicular to this surface, the former force produces no effect, the latter produces pressure against the surface. This pressure or force is again to be resolved into two component forces, one in the direction of the string of the kite, which merely stretches the string and is thus counteracted, the other force acting upwards and tending to raise the kite, which accordingly rises if the force of the wind, thus resolved, be sufficient to overcome the weight of the kite.

In like manner the action of a rudder, of the sails of a windmill, of a boat swinging across a river when moored ob-obliquely by a single rope, offer instances of effects of the resolution of force.

Section III.

The Inclined Plane.

47. The effect which the Inclined Plane AC, fig. 28, exerts to support a body W resting upon it, is the same as would be exerted by a string WR perpendicular to the plane, and fastened to a fixed point R; or by a rod WR perpendicular to the plane and moveable about a center R.

The effect of the plane is to prevent the body W from moving in any direction except that of the plane CA. The effect of the string, or the rod, would be to constrain the weight W to move in a circle of which R is the center, and to which CA is a tangent at W; and this circle and the plane have the same direction at W. Thus the effect of the plane, in preventing the motion of W, is the same as the effect of the string or rod WR.

48. Prop. *To determine the relation of the power and weight on the Inclined Plane.*

Let the power P act in the direction WP, fig. 28, and by last Article let the body be supported by a string WR perpendicular to the plane, instead of by the plane itself. Then the weight W is kept at rest by two forces in the directions WP, WR, and the proportion of these forces will be found, as in last Section, by drawing RS parallel to WP, meeting the vertical line WS in S: we shall then have (Art. 45, Cor. 1)

$$P : W :: RS : SW.$$

Let KN be drawn vertical, meeting WP in K and RW in N; then by similar triangles WRS, NWK, we have

$$P : W :: WK : KN.$$

Cor. 1. If the force act parallel to the plane, we have

$$P : W :: WC : CN;$$

and by similar triangles $WC : CN :: BC : AC$; therefore

$$P : W :: BC : AC;$$

or power . weight :: height of plane : length of plane

as was proved in Art. 44.

Cor. 2. If the force act horizontally, draw WD parallel to AB; and

$$P : W :: WD : DN :: BC : AB,$$

by similar triangles;

or, power : weight :: height of plane : base of plane.

Equality of Momenta of the Power and Weight.

49. Prop. *In the inclined plane, when the weight is supported by a force parallel to the plane, if a small motion be given to the weight, the force is to the weight as the weight's velocity, in the direction of gravity, is to the power's velocity.*

Fig. 49. Let W ascend through Ww, then P descends through an equal space Pp. And Wm being horizontal, wm is the space ascended through by W in the direction of gravity. Hence, W's velocity in the direction of gravity : P's velocity :: $wm : Pp :: wm : Ww :: BC : AC :: P : W$.

Cor. 1. Hence $P \times P$'s velocity $= W \times W$'s velocity in the direction of gravity. Therefore the momenta of the power and weight *in the direction of gravity* are equal.

Cor. 2. In like manner, if the power act parallel to the base of the plane, and if a small motion be given to the weight, we shall find that the momenta of the power in its own direction and of the weight in the direction of gravity are equal.

Hence it appears that when a weight is raised along the inclined plane, by forces acting in the manner just supposed, the velocity of vertical ascent becomes less in the same proportion in which the weight which can be supported becomes greater.

Hence, if the weight is to be raised through a given height by such forces, what is gained in power, by using the inclined plane, is lost in time.

It may be observed here, as was observed in speaking of the lever, that if we find the force which will support a weight, any greater force will raise it.

Examples of the Inclined Plane.

50. Large stones and other large masses are raised through small heights by means of inclined planes. On rail roads, inclined planes are used in order that carriages may ascend and descend from one level to another; the requisite power being supplied by a stationary steam engine, or other engine. Inclined planes of wood, smeared with soft unctuous substances, are used in the launching of ships, the vessel sliding down these planes into the water so as to avoid the violence of a vertical descent. All the improvements in roads tend to diminish the angles of the inclined planes which they offer, and thus to diminish the force requisite to draw weights up them. In ascending a steep hill the waggoner winds from side to side of the road, by which he in effect diminishes the inclination of the plane.

Section IV.

The Wedge.

51. The Wedge is an inclined plane, moveable in the direction of its base: or two inclined planes with a common base, and moveable in the direction of this base. It is employed to raise or separate obstacles, which can move only in a given direction.

In fig. 31, CAc represents a wedge, composed of two inclined planes ACD, AcD; when this wedge is thrust forwards in the direction DA, the points W, w are made to move in the directions WU, wu.

As the simplest case, suppose that the obstacle can move only in the direction of the line *WU*, perpendicular to the line *DA* in which the force acts. Then the force which urges the wedge being in the direction *DA*, the effect will be the same whether the wedge is pushed forwards, or the obstacle drawn back by a force parallel to *AD*, the base of the inclined plane *ACD*. Hence, by Cor. 2, Art. 48,

$$P : W :: DC : AD.$$

Cor. In this case, as in the inclined plane, Art. 49,

$$P : W :: W\text{'s velocity} : P\text{'s velocity};$$

the velocities being estimated in the directions in which the forces *P* and *W* act.

Section V.

The Screw.

52. The Screw is a solid cylinder, on the surface of which a spiral thread is cut, and which turns in a hollow cylinder, on the surface of which is cut a spiral thread fitting the other spiral thread. Hence, if one of these cylinders be fixed and the other turn round, the moveable one will, by turning, advance in the direction of its length.

The force employed to *turn* the screw is the power; the force or resistance which it can exert *in the direction of the length*, is the weight.

In fig. 30, *FGH* represents the spiral thread; and the screw is supposed to be turned by a force acting in the direction *MP* and turning the cylinder by a lever *CM*, *C* being a point in the axis of the cylinder. The effect produced is supposed to be that of supporting the weight *W*.

The spiral thread of the screw *FGH* may be considered as an inclined plane *fnh* wrapped round the cylinder, the base of the plane *ff'* being the circumference of the cylinder, and the height of the plane *hf'* being *HF* the distance of two threads of the spiral. By the revolution of the screw this inclined

plane will move in the direction of the base, and will act in the same manner as the wedge.

Let P, fig. 30, be the force which turns the screw; and let it act parallel to the base of the cylinder by means of a lever CM of which the length measured from the axis of the cylinder is l, and let r be the radius of the base of the cylinder. Then the force P produces at the thread a force also parallel to the base of the cylinder, or of the inclined plane, which force we will call Q; and this force Q acting parallel to the base of the inclined plane produces a force in the direction of the height of the plane, or of the length of the cylinder, which force we will call W. We then have by the properties of the inclined plane, Art 48,

$$P : Q :: r : l,$$

that is :: circumference to rad. r : circumference to rad. l,

or $P : Q$ :: circumference of cylinder : circle described by P.

Also, as before, $Q : W$ :: height of plane : base of plane or circumference of cylinder.

Hence, compounding these proportions, $P : W$ :: height of plane or interval of threads : circle described by P.

Cor. In one revolution of the power P, the screw advances longitudinally by a space equal to the interval of the threads. Hence,

$$P : W :: W\text{'s velocity} : P\text{'s velocity};$$

and the momenta of the power and weight are equal as before.

CHAPTER IV.

ON THE WORK DONE BY MACHINES.

Section I.

Measure of Efficiency.

53. The Mechanical agents which man employs, as animal strength, weights, water, steam, produce motion by exerting pressure. The pressure thus exerted is made to produce the desired effect by means of various mechanical contrivances, among which are the mechanical powers already described. By means of these mechanical powers the amount of pressure may, as has been seen, be altered in any degree. But the *work done* does not depend on the pressure alone. Thus if materials are to be raised from the ground to the top of a building, not only the weight raised, but the space through which it is raised requires to be taken into the account. When a large stone has been raised through a height of 10 feet and deposited on a scaffold at that height by the operation of a certain force, it will again require the same force to raise it through 10 feet more and place it on a scaffold 20 feet from the ground.

If we have a force which, by means of the pressure it exerts, will support any weight, or balance any machine, any additional force will produce motion, and will *work* the machine, provided the force be of such a nature that it continues to exert the same pressure while it is in motion as it did in the case of equilibrium.

The mechanical agents commonly employed are such that when in motion they continue to exert pressure; and the work

which they can do depends upon the pressure which they can thus exert while they are in motion. When a man turns a heavy wheel he exerts a certain pressure on the wheel when it is already in motion, and in order to keep up this pressure he has to follow its motion with his hand. His effective pressure is only the pressure he can exert while his hand is thus moving. In the case of a steam engine, steam can be generated so rapidly that the pressure is not sensibly diminished by the receding of the piston pressed. In motions produced by weights, the effect of the weight is not at all diminished by the motion of the machine, when the motion is uniform.

The work done by a machine may be represented as certain pressures exerted through certain spaces; and when the machine works uniformly, the work done will be doubled when the space through which the pressure operates is doubled. Thus the resistance which takes place during the turning of a millstone being supported to be constant, the work done in producing two turns of the stone must be supposed to be double of the work done in producing one turn. Also the work done by a force which produces a given number of turns of the stone must be greater, exactly in proportion as the resistance made in each turn is greater. Hence the work done must be measured by the resistance made in each turn, and by the space through which the stone has turned jointly. And in like manner in any other case, the work done must be measured by the pressure exerted at the work and by the space through which this pressure is exerted.

Let any machine be taken in which the rate of working is uniform; and let the pressure exerted by the moving force be supposed to be independent of the rate of working. We shall then find that the work done is also independent of the rate of working. It depends only upon the pressure exerted and upon the space through which it is exerted, and cannot be either increased or diminished by any alteration in the nature of the machine by which the work is done, or in the velocity with which the machine is made to work.

The proof of this assertion is contained in the following Proposition.

54. PROP. *If* f *be the pressure directly exerted by any moving force employed in doing work,* s *the space through which this pressure acts,* F *the pressure exerted where the work is done,* S *the space through which this pressure acts; then* f, s, F, S *being measured in numbers, we shall have*

$$f \times s = F \times S.$$

If F, f represent pressures which *balance* each other on any machine; S, s, the spaces through which the points, at which these pressures are exerted, move by any motion of the machine; it has been shewn in the cases of the lever, the inclined plane, and the screw, that the momenta of the power and weight are equal, (Articles 32, 49, 52) and hence in these cases the equation $f \times s = F \times S$ is true. The same equation is true of all other machines and combinations of machines, as may be proved nearly in the same manner.

Now if f represent the pressure exerted by a moving force *doing work*, the equation $f \times s = F \times S$ will still be true for the value of f which would balance F; and any addition made to f will overcome the resistance of F, and will *work* the machine.

Also, all work done by a machine may be represented as certain pressures, exerted at certain points, through certain spaces; therefore the work done in any machine may be expressed by means of F and S; and we may neglect the small addition which must be made to f in order to work the machine. Therefore, for any machine, doing work, we have the equation $f \times s = F \times S$.

COR. The product $F \times S$ may be taken as the measure of *the work done*, according to what has been already said. And it appears that if by altering the machine, we change the value of F, S undergoes a corresponding change, so that the product $F \times S$ continues the same as before. The amount of work done, thus measured, is not affected by such a change. In all cases the work done is equal to the product $f \times s$.

It has been proposed by Mr Davies Gilbert (*Phil. Trans.* 1827, p. 25) to call the product $f \times s$ the *Efficiency* of the force

f. If we adopt this term, the above proposition may be thus expressed:

The work done by any machine is always equal to the Efficiency of the moving force; it is not increased or diminished by any intermediate machinery, nor by any rate of working which does not diminish the pressure exerted by the moving force.

But machinery may very much increase the convenience of application of the force, and may modify the time in which it produces its effect; and entirely determines the kind of the work done.

55. If we express the Efficiency of any mechanical agent in numbers, according to the measure explained in the last Article, we have the measure of the work which this agent, in the given circumstances, may be made to perform.

Let it be supposed that we have weights amounting to 100 pounds at the top of a building 100 feet high. These weights, if deprived of support, have the power of descending 100 feet, either in one mass or separately. If they are attached to a machine, as powers, they may in their descent put the machine in motion and move it till they reach the ground, and may during this period, raise weights or produce pressures of suitable magnitude.

If weights be expressed in pounds and spaces in feet, the whole amount of work thus done cannot exceed 100×100 or 10000.

Thus the weights in their original position may be made, by means of proper machinery, to raise 10000 pounds 1 foot high, or 1 pound 10000 feet high.

And the efficiency is the same whether the whole weight descend at once, or in parts; and whether it descend the whole height at once, or descend a small height at a time.

When any portion of it has descended the whole or part of the height, the efficiency is so much diminished. Thus, if one quarter of the 100 pounds have descended to the ground, and one quarter to the height of 40 feet, one quarter remaining at the height of 80 feet, and one quarter at the top; the remaining efficiency is

$$25 \times 40 + 25 \times 80 + 25 \times 100 = 5500.$$

The same is true of a fluid. Any number of pounds of water at a certain height above the point to which it can fall, has an efficiency which may be estimated in the same manner which we have employed in the other case. And this efficiency may be brought into action by causing it to turn a water wheel, or in other ways.

A cubic foot of water is 62 pounds, nearly. Hence we know the efficiency of a given supply of water.

A mill dam is 100 feet square, and the mean fall is 10 feet; what depth of water in the dam is requisite to produce an efficiency of 200000?

One foot depth in the dam gives 10000 cubic feet, which is 620000 pounds. Hence the depth for an efficiency of 200000 is $\frac{200000}{620000}$ or $\frac{20}{62}$ or $\frac{10}{31}$ feet, or 4 inches nearly, supposing no force lost.

In like manner, when workmen mount a ladder, their efficiency may be estimated by the weight which they raise, (including that of their own bodies,) multiplied into the height through which they ascend. And this efficiency might be brought into action by causing the men to act by their weight in descending.

56. In like manner the efficiency of any other agent, as steam, may be measured. Let a given quantity of steam be sufficient to fill the cylinder of a steam engine, and to drive before it the piston, producing a certain pressure upon each square foot of the surface of the piston. The pressure on the whole surface of the piston is thus known; and this pressure multiplied into the length of the stroke, is the efficacy of the given quantity of steam.

It appears that the quantity of steam of a given tension produced by a certain quantity of coals, is nearly the same, whether the steam be produced more or less rapidly, if the heat be carefully economized. Hence the efficiency may be attributed to the coals, and we may consider the mechanical efficiency of a bushel of coals as constant, under all judicious modes of application.

Section II.

Moving Powers.

57. We shall here enumerate the principal sources of moving power with which we are acquainted. In all cases their Efficiency may be estimated in the manner above mentioned.

1. *Animal Force.*

The force of men is used to produce motion in a vast variety of instances: it is exemplified by all who lift weights or apply tools by their unassisted strength, or by levers, pullies, or other machines. Their own muscular force is the moving power employed by the sailor, the blacksmith, the woodsman, the workman at the printing press, and innumerable others. The grindstone, the pump, the lathe, the oar, are so moved. The processes of dragging a weight, of overturning a body, of slinging, throwing, hammering, are also exemplifications of the same original agency. Indeed, all our common muscular actions may be considered as exertions, for our own purposes, of a certain portion of the animal force which we possess; the boy who spins his top, the man who winds up a watch or a clock, exert a temporary force which, by certain contrivances, produces a more durable motion. The force of our muscles used in walking, running, leaping, produces its effect directly. We exert our muscular force for other useful purposes almost without being conscious of it, as in eating, speaking, breathing: and in other cases this force still works for our use or preservation, without our being accessary to the operation; as when the muscular force of the heart produces the circulation of the blood.

The muscular force of animals is often used; as in the case of horses drawing carriages or boats, or turning mills and wheels of various kinds. Dogs too, as in the case of the turnspit, are used to supply moving power; and in various countries, various animals take the office of beasts of burthen.

2. *Elasticity of Springs.*

The elasticity of springs is in many cases used to produce continued motion; as in the watch, the musical snuff box, the spring jack, and various automata: sometimes to produce a more sudden and intense effect, as in the case of the bow, and the trap. But the *efficiency* of this source of motion, the amount of work done by it, is comparatively small.

3. *Elasticity or pressure of aerial Fluids.*

Steam, the most prominent example of this kind, is important from the extent of its employment. It is employed either to raise a piston which the pressure of the air afterwards pushes down, and thus does the work; or the piston is pushed both ways by the elasticity of the steam, the elastic fluid being drawn alternately out of each end. Air guns act by the sudden expansion, fire arms by the sudden generation of air. The disturbance of the equilibrium of air, by the change of temperature or other causes, produces currents which are used as sources of motion, as in the smoke jack; and on a larger scale in the case of the winds, which exert their forces on the windmill, and on the innumerable sails of ships with which the surface of the ocean is everywhere speckled.

4. *Weight and Gravity.*

This is a very extensive source of motion, as in the clock, in which it moves both the pendulum and the weight: in the heavy hammer, in which gravity is often combined with other forces; in the pile-driver, where gravity alone produces the impact.

But the most durable motions produced by gravity are those in which it acts on water, as in the river, which derives its velocity from its descent whether the slope be perceptible or not, the waterfall, the mill race. The *undershot* wheel is moved by the pressure of the stream, the *overshot* wheel by the weight of the water poured into the floats on one side. The motion of the flying bridge, which moves from one side of the river to the other, by the force of the stream, is referable to such agency.

5. *Attraction.*

This force includes the preceding one, weight, as a particular case, but it exerts little influence on earth, except as it appears in that case. Motions are produced by magnetic, electric, and thermo-electric action, but they are few, slight, and transient.

But among the celestial bodies attraction is a constant and inexhaustible source of motion. The perpetual revolutions of the planets, moon, and the earth, are all owing to this universal and uninterrupted agency.

The surface of the ocean is twice every day raised through several feet by the attraction of the sun and moon. The *efficiency* thus produced is immense, and it might be employed *to do work* for man. Tide-mills on this principle have, in some cases, been constructed.

These are all the sources of moving power which we know of. Without the operation of one or other of them, not the slightest particle of matter can be put in motion. And the work done is always exactly equivalent to the force exerted.

We shall consider the measure of such work more particularly in the case of the steam engine.

Section III.

The Steam Engine.

58. Prop. *To find the theoretical Efficiency, or theoretical* Duty *of an atmospheric Steam Engine.*

The cylinder of the steam engine being filled with steam, and this steam being suddenly condensed, the piston is driven by the force of the atmosphere on the outside, and this force is the moving power of the engine.

The theoretical Efficiency of this moving power will be estimated, according to the preceding principles, by the pressure multiplied into the space through which it acts.

If the pressure be estimated in pounds, and the space through which it is exerted in feet, the efficiency habitually exerted by the machine for available purposes, corresponding to the consumption of one bushel of coals as fuel, is called the *Duty* of the machine.

The density of the air being such that the barometer stands at 30 inches, it appears, by repeated experiments, that one bushel of coals will convert 14 cubic feet of water into steam of the same elasticity as the atmosphere. Such steam occupies 1330 times more space than the water from which it was produced; that is, 14×1330 or 18620 cubic feet. Hence it would fill a space of 18620 feet in area by one foot high, so as to balance the pressure of the atmosphere. And when this volume of steam is condensed, so as to leave this space empty, the pressure which was balanced will produce motion through the one foot of height thus left unoccupied. Therefore the efficiency of the power produced will be that of the atmospheric pressure on 18620 cubic feet, acting through a space of one foot.

Now, the pressure of the atmosphere is equal to that of a column of 30 inches of mercury, which at the specific gravity of 13,568, is equal in pressure to a column of water 33,92 feet in length. Therefore, the atmospheric pressure on 18620 cubic feet is the weight of $33{,}92 \times 18620$ feet of water; and the efficiency of one bushel of coals is the weight of $33{,}92 \times 18620$ feet of water, acting through a space of 1 foot.

The weight of a cubic foot of water is 61,78 pounds. Hence the efficiency of one bushel of coals, used in the steam engine, is

$$33{,}92 \times 18620 \times 61{,}78 = 39361000 \text{ pounds.}$$

This is the theoretical efficiency: but considerable deductions must be made to obtain the practical duty for available purposes.

A portion of the heat is wasted.

The vacuum is not complete, and there is a resistance arising from the uncondensed steam.

An airpump is used to draw out the uncondensed steam, and this is of such a size as to absorb about one eighth of the efficiency.

The friction is very considerable, and diminishes the efficiency.

To estimate the available efficiency, we must deduct the part employed in moving the machinery itself.

Machines which are wrought by the expansive force of steam instead of the pressure of the atmosphere, require a separate and different calculation.

59. Prop. *To find the actual Duty of a Lifting Engine from the work done.*

When an engine is employed in lifting water by means of pumps, the available efficiency of one bushel of coals will be found by taking the product of the number of pounds raised, and of the number of feet through which they are raised, and dividing by the number of bushels of coal consumed in the same time.

The quantity of water raised at each stroke is the whole column in the pumps, and the weight of this, multiplied by the length of the stroke, and by the number of strokes in a given time, gives the efficiency for that time.

Let the column of water be cylindrical: and let D be its diameter in inches: C the height of the column in fathoms. Then the weight of the column is as D^2C; and when D and C are each 1, this weight becomes the weight of a cylinder of water of 1 inch diameter and 6 feet long, which is 2 pounds nearly. Hence $2 \times D^2C$ is the weight of the column in pounds.

Also, if L be the length of the stroke and N the number of strokes in a given time, $2 \times D^2CLN$ is the efficiency in that time.

And, if B be the number of bushels of coals used in the same time, $\dfrac{2D^2CLN}{B}$ is the *Duty* of the engine.

Ex. An engine works pumps of the depth of 58 fathoms, 17 inches in diameter, at the rate of 6 strokes, of $5\frac{1}{2}$ feet long each, in a minute. It consumes 14080 bushels of coals in 61 days: to find the duty.

$L = 6$ feet: and in 61 days $N = 5\frac{1}{2} \times 60 \times 24 \times 61$.

$$\text{Hence, duty} = \frac{2 \times 17^2 \times 58 \times 6 \times 5\frac{1}{2} \times 60 \times 24 \times 61}{14080}$$

$$= 6901753.$$

60. This is a calculation made for a standard engine in 1778. In the year 1793, after Boulton and Watt's improvements in the steam engine, it appeared that the average duty of 17 engines of their construction was 19569000. The agreement on which Messrs. Boulton and Watt constructed their engines was, that they should receive one third of the saving in fuel, estimated by a comparison with the above standard engine. In the above case, the work done was increased in the ratio of 28 to 10. So that work requiring, by the old engines, 28 bushels of coal, would be performed by Mr. Watt's steam engine, by 10 bushels, and consequently 18 would be saved. One third part of these must have been paid to Mesers. Boulton and Watt as patentees, leaving a clear gain to the persons using the engine of 12 bushels, being more than the quantity consumed.

In more recent times additional improvements have been introduced, and the duty of steam engines has been much augmented. The principles, indeed, and even the mechanism of Mr. Watt's engines, have remained unaltered since their first introduction, unless a change in the precise times of opening and shutting the valves could be considered as a variation. But to such an extent has the economy of fuel been carried—by the use of steam at a high degree of temperature and consequently of pressure, usually from fifty to sixty inches of mercury above the atmosphere—by making small fire places with sharper drafts, in iron tubes surrounded by the water of the boiler,—by more effectually preventing the escape of heat,—by enlarging the engines themselves,—and perhaps by executing the work with superior accuracy, that in the monthly return of duty performed in Cornwall by the steam engines in December 1829, the best engine, with a cylinder of 80 inches, performed 75628000, exceeding the duty performed in 1795 as 3,86 to 1, and exceeding the atmospheric engines of 1778 as 11 to 1. The duty is, however, still subject to a great variation between different engines, apparently similar in all respects, the average being about forty one millions and a half.

61. Taking the above estimates, we shall have no difficulty in verifying the subjoined statements of Sir J. Herschel.

"It is well known to modern engineers, that *there is virtue* in a bushel of coals properly consumed, to raise seventy millions of pounds weight a foot high. This is actually the *average* effect of an engine at this moment working in Cornwall*. Let us pause a moment, and consider what this is equivalent to in matters of practice.

The ascent of Mont Blanc from the valley of Chamouni is considered, and with justice, as the most toilsome feat that a strong man can execute in two days. The combustion of two pounds of coal would place him on the summit†.

The Menai Bridge, one of the most stupendous works of art that has been raised by man in modern ages, consists of a mass of iron, not less than four millions of pounds in weight, suspended at a medium height of about 120 feet above the sea. The consumption of seven bushels of coal would suffice to raise it to the place where it hangs.

The great pyramid of Egypt is composed of granite. It is 700 feet in the side of its base, and 500 in perpendicular height, and stands on eleven acres of ground. Its weight, is therefore, 12760 millions of pounds, at a medium height of 125 feet; consequently it would be raised by the effort of about 630 chaldrons of coal, a quantity consumed in some founderies in a week.

The annual consumption of coal in London is estimated at 1500000 chaldrons. The effort of this quantity would suffice to raise a cubical block of marble, 2200 feet in the side, through a space equal to its own height, or to pile one such mountain upon another. The Monte Nuovo, near Pozzuoli, (which was erupted in a single night by volcanic fire,) might have been raised by such an effort, from a depth of 40000 feet, or about eight miles.

* The Engine at Huel Towan. See Mr. Henwood's Statement "of the performance of steam-engines in Cornwall for April, May, and June, 1829." Brewster's Journal, Oct. 1829.—The *highest* monthly average of this engine extends to 79 millions of pounds.

† However, this is not quite a fair statement; a man's daily labour is about 4 lbs. of coals. The extreme toil of this ascent arises from other obvious causes than the mere height.

It will be observed, that, in the above statement, the inherent power of fuel is, of necessity, greatly under-rated. It is not pretended by engineers that the economy of fuel is yet pushed to its utmost limit, or that the whole effective power is obtained in any application of fire yet devised; so that were we to say 100 millions instead of 70, we should probably be nearer the truth."

Estimate of Living Forces.

62. In the preceding calculations we have taken no account of the time employed. If the rate of working continue the same the work done increases as the time, and the fuel consumed increases in the same proportion; so that the estimate of the Duty remains unaltered.

But in comparing the work of a machine with that of a living agent (a horse or a man) it is necessary to take into account the time employed: for the work done by the living agent depends on the time.

A horse can raise 33000 pounds one foot high in one minute.

Prop. *To find the theoretical Efficiency of a Steam Engine working for one minute.*

The pressure on the piston multiplied into the space decribed in one minute, will by the preceding principles, be the Efficiency in one minute.

The pressure on the piston will be as the square of the piston, in inches, and as the pressure on one circular inch. If the force employed be that of the atmosphere, the available part of this pressure is 5,9 pounds. Hence the pressure on any piston may be found.

The space described by the piston in one minute, by the action of the force of the atmosphere, will be the length of the stroke, multiplied into the number of double strokes made per minute.

Ex. The diameter of the cylinder of a steam engine was 72 inches, and the length of the stroke 9 feet, 9 strokes being made per minute.

In this case the pressure on the piston $= 72^2 \times 5{,}9$ pounds, and the space described in a minute $= 9 \times 9$. Hence the efficiency per minute is $72^2 \times 5{,}9 \times 81 = 2477433{,}6$.

Dividing by 33000 we have 75 for the number of horses' power to which the efficiency of the machine is equivalent.

Section IV.

Action and Reaction.

63. Prop. *In all cases of equilibrium the Action at each point is accompanied by a Direct Reaction equal and opposite to the action.*

By Action and Reaction are here meant pressures, or forces which can counteract each other's effects, so that equilibrium may be produced. That each such force acting on any point of a machine, must be accompanied and counteracted by an equal and opposite force, follows from the nature of equilibrium. For one of the forces acting alone would produce motion in the point on which it acts; it must be balanced by the other force; and from the circumstance of the two directly balancing each other, each is measured by and is equal to the other.

Such Reaction may be conceived to exist at the same point at which the action takes place, in a direction exactly opposite; or the two forces may act at two points of a line which coincides with the direction of both; as when two equal and opposite forces pull at the two ends of a rope, or push at the two ends of a stiff rod.

Cor. The effect of the Direct Reaction may, by the nature of the machine be *transferred* to some other point of the machine, where it may act at a mechanical advantage, as will be seen in the examples of a boat impelled by oars, or of a Locomotive Engine.

If, either in consequence of this circumstance, or in any other way, there exist an action not counteracted by an equal and opposite Reaction, motion will be produced.

64. This proposition will be illustrated by the following *Examples.*

A man standing on the ground produces on the ground a pressure or action equal to the weight of his body; and is supported by reaction of equal magnitude.

If he take an additional weight into his hand, the action on the ground, and consequently the reaction, are immediately increased.

If a weight hang by a rope from the ceiling, the force which stretches the rope, or the *tension* of the rope, is equal to the weight of the body.

If the rope, having one end fastened to a fixed point, pass horizontally, (or in any other direction) over a fixed pully, the other end hanging down with a weight appended to it, the tension at every point of the rope is still equal to the appended weight.

If the rope, having two equal weights appended, one at each end, pass horizontally over two pullies, the tension is equal only to one of the weights.

If a person hang by his arms from a rope fixed to the ceiling, the muscles of his arms must exert a force equal to the weight of his body, in order that he may support himself; and he must pull with a force somewhat greater in order to raise himself.

But if the rope be fastened to his waist, and, passing over a fixed pulley in the ceiling, return to his hand, he need only exert a force of half his weight to support himself, and any greater effort will cause him to rise. For he is supported, half by the rope at his waist, and half by the reaction of the rope in his hand, the tension of the rope being the same throughout.

If a man in the bow of a boat pull a rope fastened to the stern he will not at all affect the boat's progress. The action exerted by the rope at the stern will be exactly counteracted by the reaction at the point where the man rests upon the boat.

But if a man pull at an *Oar* of which the blade rests against a fixed obstacle outside the boat and independent of it, the reaction in a direction opposite to the pull (or towards the stern) will be equal to the action exerted in pulling. But this action, when it acts upon the boat at its side, (at the rowlock) is increased

by the property of the lever; for the fixed point outside the boat may be considered as a fulcrum; and the action will thus be greater than the reaction in the proportion of the distance of the rower from the fulcrum to the distance of the rowlock from the same fulcrum. Hence the boat will be urged forwards by the excess of the action above the reaction.

If the Oar, instead of resting against an immoveable obstacle, be in the water, the extreme point of the blade will move backwards, and the resistance of the fluid thus called into action will produce the effect of an immoveable fulcrum; the reasoning just stated will still apply, and the boat will be impelled forwards.

65. The motion of *Locomotive Engines* on a road depends upon principles nearly the same as those of the progress of a rowboat. Let fig. 32 represent the hind and fore wheels of a Locomotive Carriage, resting on an inclined plane; AB being the inclined plane, C, D the centers of the wheels. Let a force act by means of the rod EF upon the crank CF which turns with the wheel CA. The force acts from the point E, which is a fixed point in the machine, and pushes the point F. It will be seen that by such a force the wheel CA is urged to roll up the inclined plane; and therefore if the force be of a proper magnitude the tendency of the carriage to descend down the inclined plane may be balanced.

The radii CA, DB are those on which the carriage is supported. Let EF meet CA in H; draw HK parallel to CE, meeting DB in K.

The force which acts in EF produces equal and opposite pressures at E and at F. The latter may be supposed to act at H. Let EH represent this pressure; then this force acting at H is equivalent (Art. 45.) to CH, KH; of which KH only is effective, CH being counteracted by the reaction of the plane. Also the reaction HE at E, is equivalent to CE, KE; of which KE diminishes the pressure at E, and CE urges forward the machine in CE. Hence the two forces which act on the machine to move it are CE, KH. Hence the effect of a pressure exerted in EF and reacting at E is to produce two equal forces acting on the lever CA at unequal distances from the point A, about

which it turns. Hence there will be an excess of force in the direction of the force CE; and this excess may be such as to counteract the tendency which the machine has, by its gravity, to descend down the inclined plane. Also the force in EF may be such as not only to balance but to overcome this resistance; and to cause the machine to move up the plane, if sliding be prevented.

When the crank is in the position Cf, Ef being above EC, let the force in Ef be a pulling force. And let Ef meet AC in h, and let hk be parallel to CE. It may be shewn in the same manner as before, that the force in fE, and the reaction at E, are equivalent to two forces hk, EC, acting on the lever ACh to turn it round A. Therefore in this case also there is an excess of force tending to make the machine advance.

Hence if the force in EF be greater than is requisite for equilibrium, and be at proper intervals a pulling and pushing force alternately, such as may be produced by the motion of the piston of a steam engine, the carriage may travel up the plane with a continued motion.

The same mode of action of a force which would thus counteract the tendency to descend on an inclined plane, might also counteract the effects of friction on a horizontal plane; and if the force were of proper magnitude, might give the machine a progressive motion.

CHAPTER V.

SPECULATIONS WHICH LED TO THE NOTION OF ACCELERATING FORCE.

SECTION I.

Introductory Attempts.

66. WHEN men began to speculate concerning the motions of bodies which fall from a considerable height, they soon observed these two facts; *first*, that a body went on moving quicker and quicker the further it fell: *second*, that heavy bodies fell more quickly than very light ones.

The first rude guesses which were made to explain these facts and to assign their laws, were, as might have been expected, erroneous. It was held by Aristotle that heavy bodies are accelerated by the air which rushes in behind them to fill up the void their progress leaves; and that large bodies fall faster than small ones in proportion to their weight.

The slightest attempts to verify these laws by experiment and by tracing them to their consequences, would have shewn them to be false. But unfortunately in the times succeeding those in which these doctrines were promulgated the exact sciences were studied only as sciences of deduction. It was supposed that the first axioms of natural philosophy were to be discovered by their own internal evidence; experiment was not appealed to suggest or to verify them; principles once asserted by eminent men were thenceforth accepted without dispute; and the business of other speculators was to deduce the consequences of such principles according to the rules of logic.

This continued to be the case for nearly two thousand years. Galileo was the first person who drew the attention of the world to the necessity of examining, by comparison with facts, the truth of the asserted laws of motion.

It was easily shewn by experiment that the second of the above laws was false. Balls of 100lbs. and 1lb. were let fall from the famous leaning tower of Pisa; and instead of falling in the same time through spaces which were as 100 to 1, it appeared that the larger anticipated the smaller in its descent to the ground by two inches only. This small difference may justly be attributed to the resistance of the air, which produces a somewhat greater effect on the lighter body. And by similar experiments it was shewn that bodies of all magnitudes fall towards the earth with equal velocities, except so far as they are affected by such causes of slight irregularity.

67. With regard to the increase of velocity of falling bodies, Galileo endeavoured, in the first place, to determine the *Law* of the acceleration. He conceived that this law must be of a simple kind, and he first conjectured it to be this, that the velocity of the motion increases in proportion to the *distance* of the body from the point where it begins to move. But he afterwards convinced himself, as the fact really is, that such a law is not only false, but impossible and inconsistent with itself. The manner in which he proves this is by saying that if the velocity acquired were proportional to the space described by the falling body, all spaces, large and small, would be described in equal times. This assertion is true, but its truth is not obvious in the sense in which the argument requires it. It is obvious if we suppose the spaces to be described with uniform velocities, but this is not the case in question. It it true also when the spaces are described by velocities uniformly accelerated; but it would require several steps to establish this proposition in a satisfactory manner.

Having convinced himself that his first conjecture was erroneous, Galileo then assumed another simple law, that the velocity of the falling body increases in proportion to the *time* from the beginning of the motion. From this law it follows (as will be seen shortly) that the spaces described from the

beginning of the motion are as the squares of the times. And this consequence of the law was verified by experiment.

His method of making these experiments is detailed in the Dialogues on Motion:—"In a rule, or rather plank of wood, about twelve yards long, half a yard broad one way, and three inches the other, we made upon the narrow side or edge a groove of little more than an inch wide: we cut it very straight, and, to make it very smooth and sleek, we glued upon it a piece of vellum, polished and smoothed as exactly as possible; and in that we let fall a very hard, round, and smooth brass ball, raising one of the ends of the plank a yard or two at pleasure above the horizontal plane. We observed, in the manner that I shall tell you presently, the time which it spent in running down, and repeated the same observation again and again to assure ourselves of the time, in which we never found any difference, no, not so much as the tenth part of one beat of the pulse. Having made and settled this experiment, we let the same ball descend through a fourth part only of the length of the groove, and found the measured time to be exactly half the former. Continuing our experiments with other portions of the length, comparing the fall through the whole with the fall through half, two thirds, three fourths, in short, with the fall through any part, we found by many hundred experiments that the spaces passed over were as the squares of the times, and that this was the case in all inclinations of the plank; during which, we also remarked that the times of descent, on different inclinations, observe accurately the proportion assigned to them farther on, and demonstrated by our author."

This agreement of the law of the spaces deduced from the assumed law of acceleration, with that given by experiment, confirms the truth of the assumption, that in motion accelerated by gravity, the velocity is as the time from the beginning of the motion.

68. This law being thus established, we are naturally led to the conviction that the *Cause* of the acceleration of the motion of falling bodies is the continued action of the force of gravity. Such a notion of the cause of acceleration is not, however, a necessary part of the above explanation of the circumstances

of falling bodies. "The cause of the acceleration of the motions of falling bodies is not" Galileo observes in treating of the properties of such motions, "a necessary part of the investigation; opinions are different. Some refer it to the approach to the center; others say that there is a certain extension of the centrical medium which closing behind the body, pushes it forwards. For the present it is enough for us to demonstrate certain properties of accelerated motion, the acceleration being according to the very simple law, that the velocity is proportional to the time. And if we find that the properties of such motion are verified by the motions of bodies descending freely, we may suppose that the assumption agrees with the laws of bodies falling freely by the action of gravity."

In order to explain the acceleration of falling bodies by the action of the force of gravity, this force is to be conceived to be of such a kind that it produces equal additions of velocity in equal times, while the body falls vertically. The velocity *added* in one second to that which the falling body before had, is exactly of the same amount as the velocity communicated in one second to the body when it begins to fall from rest. The downward motion of the body does not in any degree withdraw it from the downward action of gravity. A body falling from rest acquires a velocity, in one second, of 32 feet: and if a cannon ball were shot downwards with a velocity of 1000 feet a second, it would equally, at the end of one second, receive an accession of 32 feet to its velocity, provided the resistance of the air were removed. The conception of velocity as perpetually increased by the constant action of accelerating force will become clear by a little attention.

69. The mode just explained of conceiving the manner in which gravity produces the velocity of a falling body, implies that in acquiring its motion, a body passes through every intermediate degree of velocity from the smallest to that which it last acquires. When a body falls from rest, it begins to fall with *no* velocity; the velocity increases with the time, and in $\frac{1}{1000}$ of a second the body has only acquired $\frac{1}{1000}$ of the velocity which it has at the end of one second.

This is not only certain, but manifest upon consideration: yet there was at first considerable difficulty with regard to this assertion: and disputes took place concerning the velocity with which a body *begins to fall*. And we may judge of the original difficulty of obtaining clear notions on such subjects, by the confusion under which Descartes appears to have laboured on this point. He writes in the following manner to a correspondent:

"I have been revising my notes on Galileo, in which I have not said expressly, that falling bodies do not pass through every degree of slowness, but I said that this cannot be determined without knowing what weight is; *which comes to the same thing*. As to your example, I grant that it proves that every degree of velocity is infinitely divisible, but not that a falling body actually passes through all these divisions.—It is certain that a stone is not equally disposed to receive a new motion or increase of velocity, when it is already moving very quickly, and when it is moving slowly."

It will be seen from the last sentence, that the notion of gravity, as a force that adds equal velocities to the body whatever be its previous velocity, was far from obvious.

The laws of the motions of falling bodies, and their dependence on the constant force of gravity, are stated by means of the following definitions and axioms.

Section II.

Definitions and Measures.

1. *Definition and Measure of Velocity.*

70. Velocity is the degree in which a body moves quickly or slowly.

If a body always pass over equal spaces in equal times, it is said to move *uniformly*.

In this case the velocity is *measured* by the space described or passed over in a certain time, as for instance by the number of feet passed over in one second.

Thus if a man travel 4 miles an hour, his velocity is $\frac{4 \times 5280}{60 \times 60}$ feet, or 5,86 feet a second.

The Earth, whose diameter is 7970 miles nearly, revolves on her axis in 23 hours 56 minutes = 86164 seconds. Now the circumference of a circle is to its diameter as 22 to 7 nearly. Hence the path described by a point at the equator in one second is, in feet, $\frac{22 \times 3985 \times 5280}{7 \times 86164} = 767{,}47$ feet per second.

The Student may in a similar manner answer the following questions:

The Earth moves round the Sun in an orbit of which the radius is 95000000 miles in 365 days 6 hours. Find her velocity in her orbit.

A powder is 2 days in subsiding to the bottom of a cup 3 inches deep. Required the velocity of descent of the particles.

It follows from what has been said that in uniform motion the space described in any time is the product of the numbers which measure the velocity and the time.

71. In cases of variable velocity, the following Axiom is evident.

Axiom. *In motion perpetually accelerated the space described in any portion of time is* greater *than the space which would have been described in the same portion of time if the velocity had continued uniform from the beginning of that portion:*

And is less *than the space which would have been described in the same portion of time if the velocity had been, during the whole of that portion of time, uniformly the same as it was at the end of the portion.*

In motion perpetually retarded, the contrary is true, as is equally evident.

2. *Definition and Measure of Accelerating Force.*

72. The force of gravity accelerates the motion of a falling body, adding equal velocities in equal times. This or any other force, which thus adds equal velocities in all equal times is called a *uniform* accelerating force.

A uniform accelerating force is *measured* by the velocity communicated or added in a certain time. Thus the force of gravity, which communicates a velocity of 32 feet in one second, is represented by 32; and the force of the Earth on the Moon, which would communicate a velocity of $\frac{1}{10}$ of an inch in the same time, is represented by $\frac{1}{120}$ of a foot; the proportion is 3840 to 1, which is the inverse proportion of the squares of the distances of the Earth's surface, and of the Moon, from the Earth's center.

Accelerating forces, which are not uniform, are treated by reasoning as if they were uniform for a very small time: and thus they are measured in the same way as uniform forces.

The resistance of the air, or of any other fluid, to bodies falling through it, is a retarding force, which is of the same nature as an accelerating force.

Section III.

The Laws of Falling Bodies.

73. Prop. *The velocity of a falling body will be proportional to the time of its fall from rest.*

We have already (Art. 67.) said that this was originally ascertained by experimentally shewing the relation between the space and the time to be such as would follow from this law of the velocity. And in this manner it is collected that gravity is a uniform accelerating force; its effect in communicating velocity to a falling body being independent of the magnitude of the body, and of the velocity which the body already has.

If we begin by assuming gravity to be such a uniform accelerating force, the proposition follows from the definition of accelerating force already given.

74. Prop. *The spaces described by a body falling from rest are as the squares of the times from the beginning of the fall.*

Let 32 feet be the velocity acquired at the end of one second; then 64 feet will be the velocity at the end of two seconds; 96 feet at the end of three seconds, and so on.

Also, if we subdivide these seconds into smaller parts, the velocities will still be proportional to the times. Thus, if we take eighths of seconds as the portions of time, the velocities at the ends of the

	1st	2d	3d	8th	9th	10th	16th	17th	18th	portions
will be	4	8	12	32	36	40	64	68	72	feet respectively.

Hence, by the Axiom in Art. 71, the spaces described in in each of these portions of time respectively, are *less* than

$$4 \times \frac{1}{8}, \quad 8 \times \frac{1}{8}, \quad 12 \times \frac{1}{8} \text{ \&c.};$$

that is, than

$$\frac{1}{2}, \frac{2}{2}, \frac{3}{2} \cdots\cdots \frac{8}{2}, \frac{9}{2}, \frac{10}{2} \cdots\cdots \frac{16}{2}, \frac{17}{2}, \frac{18}{2} \cdots\cdots$$

And the whole space described in the first 8 portions, or in one second, is less than $\frac{1+2+3\ldots+8}{2}$; which, by the summation of an arithmetical series, is $= \frac{9 \times 8}{4}$ or 18 feet.

In like manner, the whole space described in the first 16 portions, or in two seconds, is less than

$$\frac{1+2+3\ldots+16}{2} = \frac{17 \times 16}{4} = 68 \text{ feet.}$$

And in 24 portions, or three seconds, the space is less than

$$\frac{1+2+3\ldots+24}{2} = \frac{25 \times 24}{4} = 150 \text{ feet.}$$

But the velocities at the beginnings of the

	1st	2d	3d	8th	9th	10th	16th	17th	18th	portions of time,
are	0,	4	8	28	32	36	60	64	68	feet respectively.

Hence, by Art. 71, the spaces described in each of these portions are *greater* respectively than

$$0 \times \frac{1}{8}, \quad 4 \times \frac{1}{8}, \quad 8 \times \frac{1}{8} \text{ \&c. or } \frac{0}{2}, \quad \frac{1}{2}, \quad \frac{2}{2} \text{ \&c.}:$$

and the space described in the first 8 portions, is greater than

$$\frac{0+1+2\ldots+7}{2} = \frac{7 \times 8}{4} = 14 \text{ feet.}$$

In like manner, in 16 and in 24 portions, the spaces are greater than

$$\frac{0+1+2\ldots+15}{2} = \frac{15 \times 16}{4} = 60 \text{ feet,}$$

$$\text{and } \frac{0+1+2\ldots+33}{2} = \frac{23 \times 24}{4} = 138 \text{ feet.}$$

Hence the spaces desribed by a falling body in 1, 2, 3 &c. seconds are respectively

less than 18, 68, 150, &c. feet

greater than 14, 60, 138, &c. feet.

Therefore 16, 64, 144, &c. feet are either exactly or nearly the spaces really described.

If instead of taking eighths of seconds, we had divided each second into 32 parts, we should have found that the spaces described are

less than $16\frac{1}{2}$, 65, $145\frac{1}{2}$, &c.

greater than $15\frac{1}{2}$, 63, $143\frac{1}{2}$, &c.

And if we were to take the portions of time still smaller, we should approach more nearly still on each side, to

16, 64, 144, &c.

and it would thus appear, that the spaces really described in 1, 2, 3 &c. seconds, can be no other than these numbers of feet.

But these numbers are

$$16 \times 1^2,\ 16 \times 2^2,\ 16 \times 3^2,\ \&c.$$

Hence the spaces described in 1, 2, 3 seconds are as 1^2, 2^2, 3^2; which was to be proved.

Cor. 1. The space described by a falling body, in any time from rest, is half the space which would be described by the last acquired velocity, continued uniform.

Thus, in one second, the last acquired velocity is 32, and the space described in one second with this velocity, would be 32.

In two seconds, the last acquired velocity is 64, and the space described in two seconds with this velocity, would be 128.

In three seconds, the last acquired velocity is 96, and the space described in three seconds with this velocity, would be 228; and so on.

And these numbers 32, 128, 288, are the doubles respectively of 16, 64, 144, the spaces described by a body falling from rest in the corresponding times.

Cor. 2. Since the spaces from rest in 1, 2, 3, 4 &c. seconds are as 1, 4, 9, 16 &c., the spaces in the 1st, 2d, 3d, 4th &c. seconds are as 1, 3, 5, 7 &c.; that is, as the succession of odd numbers.

Cor. 3. If the velocity in one second become greater or less, the space in any time will be increased or diminished in the same proportion.

75. Prop. *Bodies falling in a fluid tend to a limiting velocity.*

The retarding force of resistance goes on increasing as the velocity increases, while the accelerating force of gravity is uniformly the same. Hence a falling body, since its velocity perpetually increases, tends to acquire a velocity for which

the retarding force is equal to the accelerating force. If the body once had this velocity, it would fall with a uniform velocity with no further acceleration; the resistance of the fluid exactly counteracting the force of gravity, and the body moving uniformly as if it were acted upon neither by an accelerating nor by a retarding force.

But in fact a falling body, though its velocity would perpetually approach nearer and nearer to this velocity, would never actually attain it. The velocity with which the fluid's resistance puts a stop to further acceleration is the *limit* or *term* of the velocity of a falling body: it is called the *limiting* or *terminal velocity*.

When a heavy body falls in air, the terminal velocity is considerable; when light bodies, (as the particles of fine powders) fall in denser fluids, (as water) the terminal velocity may be very small; and in such cases the body in falling through a very small space, approaches so near to the terminal velocity, that no further increase of velocity can be perceived.

Thus, a sphere of water of the diameter of 1 foot, has 400 feet nearly for its terminal velocity in air. But a powder of the specific gravity of lead, the particles of which are spheres of the diameter of ,000000089 of an inch, would have for its terminal velocity ,03 of an inch in a second, or 1,8 inches a minute. And if the particles were a million times smaller than this, the terminal velocity would be only ,00003 inches in a second, or about $\frac{1}{10}$ of an inch in an hour.

CHAPTER VI.

SPECULATIONS WHICH LED TO THE ESTABLISHMENT OF THE SECOND LAW OF MOTION.

SECTION I.

Introductory Attempts.

76. A STONE thrown from the hand describes a curved path and soon comes to the ground. The laws of such motion presented an obvious subject of speculation. Aristotle asks* "Why does the motion cease of things cast into the air? Does this happen when the force has ceased which sent them forth? or is there an opposite force which acts against the motion? or does the fact result from the disposition to fall, and occur when this disposition is stronger than the projectile force? or is it absurd to put the question in this manner, instead of referring to the general laws of motion?"

The last clause is perhaps freely translated, but, as we have given it, it suggests the true reply to the preceding questions.

No true explanation was given of the facts which suggested these questions till a much later period. A mistaken belief concerning the nature of the motion of *projectiles*, or bodies projected, contributed for some time to mislead enquirers on this subject. In the use of "*military* projectiles," it was prescribed as a rule that, for certain distances, a gun must be directed "point blank;" that is, with its barrel in a horizontal straight line towards the point aimed at; but that for greater distances the barrel must be elevated, so as to make allowance for the fall of the bullet, which was called shooting "at random." This led to the opinion that the path of a bullet discharged from a gun was a horizontal straight line till it reached a certain distance, and that after that distance it began to descend in a curved line.

* Μηχ. προβ. λγ.

Thomas Digges, in his Treatise on the New Science of Great Artillerie (1591), remarked that the bullet has, even from the beginning, a downward motion which though insensible at first, draws it from its direct course.

Tartelea also denied that a bullet ever moves in a horizontal line; but his theory was still very erroneous; for he supposed that the bullet's path through the air is made up of an ascending and a descending straight line, connected in the middle by a circular arc.

In 1609, Galileo had considered the subject, and had satisfied himself that the motion of projectiles in a vertical direction is not affected by their motion in a horizontal direction. This principle, combined with his theory of falling bodies, led him to the true doctrine of projectiles.

Galileo's principle, having been once suggested, was supported by many circumstances in the motion of bodies projected, and was especially confirmed by the discussions which took place about that period concerning the motion of the Earth.

On the Motion of the Earth.

77. The doctrine promulgated in modern times by Copernicus, that the Earth travels round the Sun, and revolves on her own axis, led to a long series of controversies, which turned mainly upon the truth or falsehood of the supposed laws of motion, and especially of the one now under our consideration. The opponents of the Earth's motion attacked that doctrine with objections drawn from erroneous mechanical principles; but the assertors of the Copernican system, being at first ignorant of the true principles which bear upon the subject, were not fortunate in their answers to the objections.

"If the Earth, it was said, revolved so rapidly from west to east, a perpetual wind would set in from east to west, more violent than what blows in the greatest hurricanes; a stone, thrown westwards, would fly to a much greater distance than one thrown with the same force eastwards; as what moved in a direction, contrary to the motion of the Earth, would necessarily pass over a greater portion of its surface, than what,

with the same velocity, moved along with it. A ball, it was said, dropt from the mast of a ship under sail, does not fall precisely at the foot of the mast, but behind it; and in the same manner, a stone dropt from a high tower would not, upon the supposition of the Earth's motion, fall precisely at the bottom of the tower, but west of it, the Earth being, in the mean time, carried away eastward from below it. It is amusing to observe, by what subtile and metaphysical evasions the followers of Copernicus endeavoured to elude this objection, which, before the doctrine of the Composition of Motion had been explained by Galileo, was altogether unanswerable. They allowed, that a ball dropt from the mast of a ship under sail would not fall at the foot of the mast, but behind it; because the ball, they said, was no part of the ship, and because the motion of the ship was natural neither to itself nor to the ball. But the stone was a part of the earth, and the diurnal and annual revolutions of the Earth were natural to the whole, and to every part of it, and therefore to the stone. The stone, therefore, having naturally the same motion with the Earth, fell precisely at the bottom of the tower. But this answer could not satisfy the imagination, which still found it difficult to conceive how these motions could be natural to the Earth; or how a body, which had always presented itself to the senses as inert, ponderous, and averse to motion, should naturally be continually wheeling about both its own axis and the Sun, with such violent rapidity. It was, besides, argued by Tycho Brahe, upon the principles of the same philosophy, which had afforded both the objection and the answer, that even upon the supposition, that any such motion was natural to the whole body of the Earth, yet the stone, which was separated from it, could no longer be actuated by that motion. The limb, which is cut off from an animal, loses those animal motions which were natural to the whole. The branch, which is cut off from the trunk, loses that vegetative motion which is natural to the whole tree. Even the metals, minerals, and stones, which are dug out from the bosom of the Earth, lose those motions which occasioned their production and encrease, and which were natural to them in their original state. Though the diurnal and annual motion of the Earth, therefore, had been natural to them while they

were contained in its bosom; it could no longer be so when they were separated from it."

"The objection to the system of Copernicus, which was drawn from the nature of motion, and that was most insisted on by Tycho Brahe, was at last fully answered by Galileo; not, however, till about thirty years after the death of Tycho, and about a hundred after that of Copernicus. It was then that Galileo, by explaining the nature of the composition of motion, by showing, both from reason and experience, that a ball dropt from the mast of a ship under sail would fall precisely at the foot of the mast, and by rendering this doctrine, from a great number of other instances, quite familiar to the imagination, took off, perhaps, the principal objection which had been made to this hypothesis."

The disciples of the school of Galileo went on confirming this view of the matter. Thus Gassendi, in his treatise "De motu impresso a motore translato" shews in a variety of ways, that a body which, while it is carried along in any vehicle, as a boat or a chariot, has another motion impressed upon it, by falling, or by being thrown, or in any other manner, retains still the motion of the vehicle. He thus refutes the objections which had been brought against the motion of the Earth by various persons, and especially by Morinus, in a treatise entitled "*Alæ Terræ Fractæ.*"

In this manner it was now seen that a stone falling from the top of a tower, ought not to be left behind by the motion of the Earth's surface from west to east, and thus to fall to the west, as had been asserted to be the consequence of the laws of nature. The stone would partake of the motion which the tower had, and would therefore, relatively to the tower, fall in a vertical straight line.

78. After it had ceased to be a tenable argument against the rotatory motion of the Earth, that the stone did not fall to the *west* of the vertical, it was asserted that a real objection was to be found in the circumstance, that the stone did not fall to the *east* of the vertical. For the horizontal velocity, from west to east, which the stone has when it is let fall, and which it retains during its fall, is that which belongs to the *top* of

the tower. But the top of the tower moves faster than the bottom by the rotatory motion of the Earth, being farther from the center. Hence, the stone ought to move farther to the east in the time of its fall, than the bottom of the tower does; and thus ought to get the start of the tower, and fall to the eastward of its base.

The answer to this objection is, that the stone really does fall to the eastward of the foot of the vertical, but that in all experiments which we can make, the interval is too small to be certainly determined by experiment, as appears by calculating its magnitude. In some experiments made in Italy, it is said that such a deviation was really detected.

By experiments and controversies of this kind, the Copernican system was finally established as the true system of the universe. The true law of nature, with regard to such cases as have just been spoken of, is stated in the Second Law of Motion.

Section II.

The Second Law of Motion.

79. *When any force acts upon a body in motion, the change of motion which it produces is in the direction and proportional to the magnitude of the force which acts.*

This may also be thus expressed. When any force is exerted upon a body already in motion, the motion which the force would produce in a body at rest, is compounded with the previous motion, in such a way, that both produce their full effects parallel to their own directions.

Thus, suppose a body, considered as a point, to be moving in the direction AB, fig. 34, with such a velocity that it may describe AB uniformly in one second of time. Then by the first law of motion it would in the next second of time describe Bb, in the same straight line, equal to AB. But when it comes to B, let a force in the direction BM begin to act and act uni-

formly upon it for one second; the force being of such a magnitude that it would in one second cause the body to describe *BM* from rest. Then at the end of one second from the time when the body is at *B*, it will be found at *C*, so that *MC* and *bC* are equal and parallel to *Bb* and *BM*.

If when the body comes to *C* the force were to cease to act, it would go on moving in the direction and with the velocity which it has at *C*. Let *Cc* be the space it would thus describe in one second. But now suppose a force to begin to act at *C*, which by its uniform action for one second would carry the body through *CN*. Then its place at the end of one second from *C*, will be *D*, *DN* and *Dc* being parallel and equal to *cC* and *NC*.

Similarly, if other forces act uniformly for successive seconds, we may find their effects. If the forces be not such that they can be considered as uniform in magnitude and direction for one second at a time, we must apply this law of motion to them for any small time during which they may be considered as uniform. If they vary continuously, we must consider the *Limits* of *Bb* and *BM*, &c. as will be seen hereafter.

The proofs on which this law rests have been partly stated in giving its history. We shall present some additional considerations.

The Composition of Velocities.

80. Prop. *When two velocities are combined, if separately they be represented in magnitude and direction by the two sides of a parallelogram, when conjoined they will be expressed by the diagonal.*

Let *PQ*, fig. 33, be a plane, as the deck of a ship, which moves parallel to itself, with a uniform motion, from the position *PQ* to the position *pq*. Let a body have, on this plane, a uniform motion, which would carry the body through *BD*, while the point *B* of the plane moves through *Bb*. If the parallelogram *Db* be completed and the diagonal *Bd* drawn, the body will describe the diagonal *Bd* uniformly by the composition of the two motions.

When B comes to b, BD comes to the position bd, and therefore the body will have moved from B to d. Also, at any intermediate time, let BD have come into the position $\beta\delta$, parallel to BD; and take $\beta\gamma : \beta\delta$ or bd :: time in $B\beta$: time in Bb; that is :: $B\beta : Bb$, because the motions are uniform.

But since $\beta\delta : bd :: B\beta : Bb$, $B\gamma\delta$ is a straight line. Also $B\delta : Bd :: B\beta : B\delta$ :: time in $B\beta$: time in Bb; hence the motion in Bd is uniform.

We have here supposed that the moving point has and retains the two velocities; it retains the velocity represented by Bb, because it is carried along with the plane, and it has the velocity represented by BD, with which it moves on the plane, and relatively to it.

In the cases which come under the second law of motion we do not suppose the body to retain the original motion and the additional motion by means of a material connexion, such as the moving plane PQ is here supposed to supply: after the additional motion is impressed, the body is left entirely to itself.

81. If a body were moving in the direction and with the velocity Bb, and if a velocity BD were impressed upon it at B, and it were then left to itself, it would, by the second Law of Motion, move in the direction and with the velocity Bd.

And in this case the body's motion, relatively to the moving space PQ, would be represented by BD.

Thus it appears, as a consequence of the second Law of Motion, that if a body, which is moving along with a moving space, have any velocity impressed upon it, the motion of the body, relative to the space, will be the same as if the body and the space had been originally at rest.

Hence, if this second Law of Motion be true, all the mechanical actions, which take place in a space moving uniformly, will be the same, relatively to the parts of this space, as if the space were at rest.

Hence this law is confirmed by our finding that the relative motions and actions of bodies, in a space which moves uniformly, are exactly the same as in a space at rest.

Thus in a ship under way, a ball will go equal distances when thrown with equal force, whether towards the bow or the

stern. The effects of the mutual pressures and impacts of bodies in such a case are the same in every direction. Also the motions of bodies on land are the same, under the same conditions, whether they take place east, west, north, south, or in any other direction, although in some of these cases the Earth's motion conspires with, in others is transverse or opposite to the motion of the bodies. The oscillations of a pendulum are performed in the same time whether they take place east and west, or north and south. It may be shewn that a very small deviation from exactitude in the asserted law would produce a perceptible difference in the last experiment.

"A man continuing to throw upwards a ball or orange, or several of them at once, and to catch and return them alternately, uses no difference of art as regards them, whether he be standing on the earth and whirling with it, or on a sailing ship's deck, or in a moving carriage, or on a galloping horse's back. He and the oranges have always the same forward common motion. And when a man, standing on a galloping horse, leaps through a hoop held across his course, he does not leap forward—for this would throw him over the horse's ears—but merely jumps up, and allows his motal inertia to carry him through."

"The reason that a lofty spire or obelisk stands more securely on the earth than a pillar stands on the bottom of a moving waggon, is not that the earth is more at rest than the waggon, but that its motion is uniform.—Were the present rotation of our globe to be arrested but for a moment, imperial London, with its thousand spires and turrets, would be swept from its valley towards the eastern ocean, just as loose snow is swept away by a gust of wind."

82. Besides the experimental proofs of the law thus given, it is naturally suggested, though not demonstrated, by its analogy with the composition of forces, and with the composition of velocities.

The object of the second Law of Motion is to give a rule for the combination of the effect of a *force* acting on a body with the *velocity* which the body already possesses.

It has been shewn (Art. 45.) that *two forces* acting in the direction of the sides of a parallelogram, and proportional to these sides, are equivalent to a force represented in magnitude and direction by the diagonal.

It has been shewn also (Art. 80.) that *two velocities* represented in magnitude and direction by two sides of a parallelogram produce, if communicated at the same time, a velocity represented in magnitude and direction by the diagonal.

Hence we might expect that when a force acts so that its effect is compounded with a previous velocity, the same rule of composition should obtain, as we find that it does.

83. We may observe however that this is not to be taken for a demonstration, as is sometimes done.

The combination of two velocities represented in magnitude and direction by the two sides of a parallelogram, will produce a resulting velocity represented by the diagonal. Hence it is sometimes said, that *if* a body moving with one of these two velocities *have* the other communicated to it, it will describe the diagonal. But the object of the second law is to show that the body *will have* the second velocity communicated to it, this velocity being that which the force would produce in the body if at rest.

It is sometimes said that the action of a force parallel to the line *DB* cannot accelerate or retard the approach of the body to the line *DB*. But this is not obvious, except we suppose the body to retain the two velocities, without their interfering with one another; whereas in reality, the two velocities are confounded into a single velocity; and the manner in which this single velocity depends upon two original ones in natural occurrences, cannot be known from our definitions merely.

As the second Law of Motion cannot be proved immediately from the composition of forces, so also the composition of forces cannot be deduced from the second Law of Motion. The conditions of magnitude and direction under which pressures balance each other, are not and cannot be dependent on the laws of the motions which take place when the forces do not balance.

Section III.

Projectiles.

84. Prop. *The path of a projectile acted on by gravity will be a parabola.*

In fig. 35, let the projectile be moving horizontally at B. Then, if gravity were not to act, the body would describe equal spaces Bc, cd, de in equal successive times, in the horizontal line. But if cC, dD, eE be the vertical spaces which the body would describe by the action of gravity in the time of describing Bc, Bd, Be, the body will, by the second law of motion, be found, at the end of these times, in the points C, D, E, BC, BD, BE being parallelograms; and the path of the body $BCDE$ will be the assemblage of all the points thus found.

Now if we take any two points, as C, D, MC and ND, that is Bc and Bd, are to each other as the times of describing BC, BD. Also BM and BN, that is cC and dD, are to each other as the squares of the times of describing BC, BD (Art. 74). Therefore BM and BN are as the squares of MC and ND; and therefore the squares of MC and ND are equal to the rectangles made by BM and BN respectively with the same constant line; which is the property of the parabola.

Thus Bc, cd, de being equal lines, cC, dD, eE will be in the proportion of 1, 4, 9.

When a body ascends, acted upon by gravity, the spaces described, measured from the highest point, follow the same law as the spaces described in descending from rest. Hence the curve XAB, described by a body ascending to the highest point B, is exactly similar to the curve BCE described in descending.

If the body begin to move from any point A, being projected in the direction AT, it will describe a portion ABE of a parabola, exactly like the one already spoken of.

Section IV.

Central Forces.

85. Prop. *If a body in motion be acted on by a force constantly tending to a fixed center, it may revolve in a curve about that center.*

Let the time be divided into equal portions, and in the first portion let the body describe AB, fig. 36. By the first Law of Motion, if no force were to act on the body, it would in the second portion of time go on to c, in the same straight line, describing Bc equal to AB. But when the body comes to B, let a force tending to the centre act on it by a single instantaneous impulse, and turn the motion in the direction BC. Draw cC parallel to BS; and by the second Law of Motion, the body will describe BC in the second portion of time, C being in the plane ASB.

In like manner if a centripetal force towards S act impulsively at C, D, E, &c. at the end of equal successive portions of time, it will cause the body to describe the straight lines CD, DE, EF &c.

Let now the number of the portions of time be augmented, and their magnitude diminished, indefinitely; and the motion of the body will still be determined in the same manner.

But when we increase the number of the portions AB, BC, CD, &c. and diminish their magnitude indefinitely, the figure approaches more and more nearly to a curve. Also the force which acted interruptedly at B, C, D, E becomes more and more nearly a continuous force; and the second Law of Motion on which the above reasoning depends, still continues to be applicable.

Therefore when the force becomes a continuous force the polygon will become a curve; and it may be a curve surrounding the center S, so that the body shall revolve round S.

In this manner the planets are retained in their orbits by the attraction of the Sun, and the satellites revolve in their orbits by the attraction of their respective primary planets.

86. Prop. *A body may describe a circle by the action of a uniform force tending to the center.*

The nature of the curve in the last proposition will depend upon the relation of the velocity and the central force; and this may be such that the curve is a circle. In this case all parts of the curve being exactly alike, the force to the center will be everywhere the same.

When a body revolves in a circle it must be retained in its path by a force tending to the center, and this force may be produced in various ways. Thus if a stone be whirled round at the end of a string, the tension of the string is the centripetal force. The Moon revolves about the Earth, and the gravity of the Moon to the Earth is the force which retains her in her orbit. If a cannon ball could be projected horizontally at the Earth's surface with a velocity of about 25000 feet a second, it would revolve round the Earth, being retained by the force of gravity.

If the centripetal force cease for a moment to act on the revolving body, as for instance if the string break by which a stone is whirled round, the body immediately flies off from the center. This fact is often attributed to the action of a *centrifugal* force. In reality however there is no such force, otherwise than as a result of the first law of motion. The body tends to recede from the center, only because it tends to go on in a straight line. It flies off, only because it flies in a tangent to the circle.

87. The tendency of revolving bodies thus to recede from the center explains many phenomena.

"If a pair of common fire-tongs, suspended by a cord from the top, be made to turn by the twisting or untwisting of the cord, the legs will separate from each other with force proportioned to the speed of rotation, and will again collapse when the turning ceases. Mr Watt adapted this fact most ingeniously to the regulation of the speed of his steam-engine. His *steam-governor* may in truth be described as a pair of tongs with heavy balls at the ends, to make their opening more energetic, attached to some turning part of the machine. If the engine move with more than the assigned speed, the

balls open or fly asunder, and by a simple contrivance are made to move a valve which contracts the steam tube; on the contrary, with too slow a motion, they collapse and open the valve.

A half-formed vessel of soft clay, placed in the centre of the potter's table,—which is made to whirl, and is called his wheel,—opens out or widens merely by the centrifugal force of its sides, and thus assists the worker in giving it form.

A ball of soft clay, made to turn quickly by a spindle fixed through its centre, soon ceases to be a perfect ball. It bulges out in the middle, where the centrifugal force is great, and becomes flattened towards the ends, where the spindle issues.

This is exactly what has happened to the ball of our Earth. It has bulged out seventeen miles at the equator, in consequence of its daily rotation, and is flattened at the poles in a corresponding degree.

In the planets Jupiter and Saturn, of which the rotation is much quicker than of our Earth, the middle or equator bulges out still more—even so as to offend an eye which expects a perfect sphere.

A mass of lead that weighs one thousand pounds at our pole, weighs about five pounds less at the equator, by reason of the centrifugal force.

If the rotation of our Earth were seventeen times faster than it is, the bodies or matter at the equator would have centrifugal force equal to their gravity, and a little more velocity would cause them to fly off altogether, or to rise and form a ring round the Earth like that which surrounds Saturn."

CHAPTER VII.

SPECULATIONS WHICH LED TO THE THIRD LAW OF MOTION.

SECTION I.

Introductory Attempts.

88. IT was early seen that the effect of a body in motion was different from that of the same body at rest. A hammer will *drive* a nail by a blow, which a very great weight could not force into the wood by pressure. The earliest attempts to explain this circumstance were erroneous, as they were with regard to other mechanical facts. Writers endeavoured to refer it to the lever: thus Guidubaldi, who has already been mentioned, says that the effect of the blow depends on the length of the handle of the hammer.

When the term *momentum* was introduced, it was for a long period used with very considerable vagueness; it was employed to designate the amount of a body's mechanical action or tendency to motion, before it had been ascertained how such actions and tendencies were to be measured and compared. Thus Galileo, in his "Discorso intorno alle cose che stanno in su l'Acqua" says that "it is the force, efficacy, or virtue with which the motion moves and the body moved resists, depending not on weight only, but on the velocity, inclination, and any other cause of such virtue."

To explain the great effects of percussion compared with those of pressure was one of the purposes to which it was attempted to turn the notion of momentum; but these attempts were at first unsuccessful. Galileo arrived at the paradox that the force of percussion is infinitely greater than that of pressure: and though Borrelli and others explained some of the circumstances which had puzzled their predecessors, the subject remained one of considerable perplexity even to very recent

times. The problem of percussion, which had first suggested the notion of momentum, was one of the last to be satisfactorily solved by means of that notion.

The true laws of the effect of pressure in producing motion were more sucessfully made out in other ways, and especially by the consideration of the inclined plane.

When Galileo had discovered the laws of the motion of bodies falling freely, he extended these laws to the motion of bodies upon inclined planes, considering it to be manifest that such motions are accelerated by laws similar to those of bodies falling freely. It has been seen, accordingly, that the confirmation of these laws by observation was in fact first obtained by means of experiments upon inclined planes.

But though the *law* of acceleration is the same when a body falls freely and when it falls down an inclined plane, the *amount* of acceleration is different. Nor did Galileo's principles give him any direct means of determining the proportion of this amount.

To avoid this difficulty he assumed an auxiliary principle, namely, "that the velocity acquired in falling down all inclined planes of the same perpendicular height is the same." And this result he confirmed by direct experiment, extending the assertion not only to inclined planes, but to all curvilinear paths whatever.

Galileo's Proof of the Velocity acquired by a body falling down a Curve.

89. The experimental proof of this principle given by Galileo is very ingenious. (*Dialogo 3. delle Scienze Nuove*). It is as follows. In fig. 38, let a string AC, with a weight as C, appended, be fastened to the point A in the vertical plane ACB, so that the weight may swing in the circular arc CBD. If the weight be let fall from D, it will descend to B and rise again to C, the velocity at the lowest point B, acquired by falling down DB, exactly sufficing to carry it up to the horizontal line from which it fell. Now let a nail be fixed at E in the vertical line AB, so that on the side of D, the weight may be compelled to move in the circular arc GB of which

the center is E. Then, G being in the horizontal line DC, let the weight fall from G and it will be found that it still arises exactly to C before its velocity is extinguished: Thus proving that the velocities acquired by falling down GB and DB are exactly equal.

90. This experimental proof however did not satisfy the requisitions of mathematical logic.

"When Viviani," says Mr Drinkwater, "was studying with Galileo, he expressed his dissatisfaction at this chasm in the reasoning; the consequence of which was that Galileo, as he lay the same night, sleepless through indisposition, discovered the proof which he had long sought in vain, and introduced it into the subsequent editions."

The principle of this proof he expresses by saying that the *momentum* of the body with which it tends to descend down the inclined plane is to the momentum with which it would descend in the vertical line, as the height of the plane is to its length. When the momentum is understood to be proportional to the accelerating force, this proposition agrees with the third Law of Motion; and its application will be seen in tracing the consequences of that law.

In the course of time the notion of momentum became more precise. It was discovered that *the product of the numbers representing the weight moved and its velocity*, might in very many cases be used as a measure of the power to produce motion, and that a number of rules might thus be brought under one more general rule. It was then seen, (as we have already shewn it to be true) that by taking this definition the conditions of equilibrium of all machines were expressed by saying that "the momentum of the power and weight must be equal."

This notion of momentum was soon applied so as to give the true laws according to which motion is produced, both in the case of the inclined plane, of impact, of pendulums, and many others. At first however it was by no means clearly or philosophically expressed. Thus Wren in his paper on the Laws of Collision, begins by stating that "the *proper* and most *natural* velocities of bodies are those which are reciprocally proportional to these bodies." When bodies meet,

having such velocities, they preserve them. If they have *improper* velocities, one loses and the other gains, so that the case has an analogy with a lever supported on two centers. (*Phil. Trans.* 43).

91. The general problem on which all these particular ones depend is, to determine the velocity produced by a given pressure, knowing the weight of the body acted upon and the time during which the force acts.

When pressure produces motion, we cannot collect from any of the preceding reasonings, the velocity generated. It is obvious that a greater pressure produces a greater velocity, other things being equal: it is obvious also that the greater the mass the less is the velocity which a given pressure generates. If a small weight be suspended by a string, we can, by a small push, give it a considerable velocity; but if a large rock be suspended by a rope, the same push would scarcely produce a perceptible motion, though the friction here would be altogether inconsiderable.

But though in such a case a very small motion would be produced, there would be, in reality, *some* motion produced, however small were the force and however large the mass to be moved, if the motion were not resisted by something besides the mere *inertia* of the body, as for instance friction, or a fixed obstacle. This is sometimes illustrated by saying that a man walking or leaping on the earth's surface, moves the whole globe of the earth by the pressure of his feet. And this is mathematically true, but the velocity of the earth is as much less than that of the man, as its weight is greater; and the motion will be less than the millionth of the millionth of the millionth of a hair's breadth; that is so far the world of sense is concerned, it is the same as if it did not exist.

We may however put a very large quantity of matter in motion by a very small force, practically as well as theoretically, by continued or repeated action. Thus a large weight hung up by a string may be put in perceptible motion of the breath, if we blow at intervals answering to the oscillations of the string, so that the small impulses may all assist and not counteract each other's effects.

The simplest law which we can imagine to connect pressure and the velocity produced by it, so as to answer the above obvious conditions, is that the velocity produced by any pressure shall be greater exactly in proportion as the pressure is greater, and shall be less exactly in proportion as the mass moved is greater. This is expressed mathematically by saying, that the velocity produced *varies directly* as the pressure and *inversely* as the mass moved. It will be proved that this is the true Law of Motion if we assert it of the velocity produced *in a given time* by a uniform pressure.

If this velocity vary as the pressure directly and as the mass moved inversely, it follows from the mathematical rules of variation, that the product of the velocity and the mass moved (both being represented by numbers) varies as the pressure: that is (by the definition of momentum) *the momentum produced in a given time varies as the uniform pressure which produces it.*

92. In order to prove this rule, which we shall do in the next Section, we must trace some of its consequences.

When Pressure acts upon a body in motion, and in the direction of the motion, the velocity which is *added* must be the same as the same pressure would produce in a body at rest. And when the pressure acts in a direction opposite to the motion, the momentum which the pressure *subtracts* or *destroys* must be the same as it would produce in a body at rest. Thus if two equal pressures act, one in the direction of a body's motion and the other in the direction opposite to a body's motion one would produce as much momentum as the other destroys.

Now when pressure is exerted between two bodies, the pressing body experiences a pressure equal to that which it exerts, and in an opposite direction (Art. 78). Thus when a heavier weight draws a lighter over a single fixed pulley, the pressure of the string which connects them pulls one of the bodies forwards and the other backwards with the same force. When one body strikes against another and pushes it forwards for a time, the first body has its motion retarded by an equal push. And therefore in either case one body must lose as

much momentum as the other gains, for the equal pressures act upon them for equal times.

Let two bodies, each of the magnitude 1, be connected by a string, so that one lies upon a table and the other hangs freely over the edge of the table. The former body will have no tendency to move, but the two bodies will be put in motion by the weight of the latter. If the latter body were to fall freely, it would in one second acquire a velocity of 32 feet, and therefore a momentum of 32. In consequence of having to drag along with it the other body, the velocity will only be 16; but the momentum will still be 32, since both bodies move with the same velocity; the body which hangs loses 16 of the momentum which it would have had, and the body which lies on the table acquires a momentum 16 which it otherwise would not have had.

In this manner it is universally true, that in the direct action of two bodies in motion, *the momentum gained by one body and that lost by the other must be equal.*

The momenta thus gained and lost must be reckoned in the direction of the motion; and to produce any momentum in one direction must be conceived to be the same thing as to destroy an equal momentum in the opposite direction. Thus let bodies A and B, of which the magnitudes are 2 and 1, hang at the ends of a string over a fixed pully. A will descend and B will ascend. If they had not been connected, both would have descended; A would have acquired in one second a velocity 32 and a momentum 64; B would have acquired a velocity 32 and a momentum 32. In consequence of their connexion, A will acquire in one second a velocity only of $10\frac{2}{3}$ and a momentum of $21\frac{1}{3}$, thus losing a momentum of $42\frac{2}{3}$; while B, which would have descended with a velocity of 32, ascends with a velocity of $10\frac{2}{3}$; and thus B gains, *in the direction of the motion*, first, a momentum of 32 which destroys its descending velocity, and then an additional momentum of $10\frac{2}{3}$, with which it ascends; making thus an ascending momentum gained amounting to $42\frac{2}{3}$, the same amount which A loses.

In the same manner if it appear by the calculation that a body loses more momentum than it really possesses, the result will be that it will move in an opposite direction to that in which its momentum was.

93. The equality of pressure between two bodies which act upon each other mechanically is a universal statical truth, and is expressed by saying that "Action and Reaction are equal and in opposite directions."

The fact that such mutual pressures *produce and destroy* in the two bodies *equal momenta*, requires a distinct and separate proof. It has sometimes been expressed in the same words which assert the equality of the pressure, namely that "Action and Reaction in opposite directions are equal;" but this mode of stating the matter can only lead to ambiguity and confusion. To make Action and Reaction mean momenta gained and lost, after we have previously made the words mean pressures exerted in opposite directions, is highly illogical; and answers no purpose except that of making an experimental proof appear unnecessary, in a case where it is the only possible demonstration.

94. When the mechanical problem of the Impact of Bodies was treated, the impact was generally assumed to be instantaneous. In this way it became an action *sui generis*, and there was no reason *a priori* why the same rules should apply to this action as apply to continued pressure. It was however assumed that the same rules did apply.

The fact is that no impact is, properly speaking, instantaneous; all collision is merely a short and violent pressure: and to such pressures the laws of action and reaction are of course properly applicable. Thus an error in the hypothesis concerning the nature of impact was balanced by a groundless assumption concerning the laws which apply to this kind of mechanical action.

The laws of the impact of bodies are deduced from the principle already mentioned with regard to other cases of mechanical action: namely that the momentum gained and lost are equal: and this is strictly applicable, however the pressure be supposed to vary during the contact. For the pressure on the two bodies at each instant will be equal; and therefore the momentum gained by the one in each instant, and therefore in the whole result, equal to that lost by the other.

On this principle the principal cases of the problem of impact were solved independently by Wallis, Wren, and Huyghens in 1668.

On the Motion of the Center of Gravity.

95. At the end of the solution which Huyghens gave of the problem of impact, he stated that he had discovered and partly proved a law concerning the motion of bodies impinging on each other, which he suspected to be universal. The law was this.

In the impact of bodies, the motion of the center of gravity takes place in the same direction and with the same velocity before and after the collision.

This proposition is universally true. We shall shew, in the case of the motion of bodies in the same line, that it follows from the equality of the momenta gained and lost.

Let A, B, fig. 37, be two bodies, AC, BD spaces described by them in the same time uniformly, therefore AC, BD measure the velocities of the bodies before impact. Let G be the center of gravity when the bodies are at A and B, H the center when they are at C and D. Therefore GH is the velocity of the center of gravity.

Now A, B, being the magnitude of the bodies, we have by Art. 41, since G is the center of gravity, $A \times GA = B \times GB$

$$\text{or } A \times (GC + CA) = B(GH + HB)$$

$$\text{or } A \times GC + A \times CA = B \times GH + B \times HB,$$

Also, since H is the center of gravity when A is at C and B at D, $A \times CH = B \times DH$,

$$\text{or } A \times (GC + GH) = B \times (HB + BD),$$

$$\text{or } A \times GC + A \times GH = B \times HB + B \times BD:$$

subtracting this from

$$A \times GC + A + CA = B \times GH + B \times HB,$$

$$\text{we find } A \times CA - A \times GH = B \times GH - B \times BD,$$

$$\text{or } A \times AC + B \times BD = (A + B) \times GH,$$

or the sum of the momenta of the two bodies A, B, moving with their own velocities, is equal to the momentum which the sum $A+B$ would have if it were to move with the velocity of the center of gravity.

Now, since, (Art. 94.) the momenta gained and lost in impact are equal, the sum of the momenta of the bodies after impact is the same as it is before impact, subtracting instead of adding when the motion of either of the bodies is in the opposite direction. Hence the momentum of $A+B$, moving with the velocity of the center of gravity, remains unaltered: and therefore the velocity of the center of gravity, in its original direction, remains unaltered.

This is true whether the bodies be elastic or inelastic.

Upon the same principles it might be proved if the bodies were to meet each other obliquely in any manner whatever, that the motion of the center of gravity is not affected by their mutual impacts and pressures.

We now proceed to state the third Law of Motion, previously defining the notions which it involves.

Section II.

The Third Law of Motion.

96. The third Law of Motion depends upon the Quantity of Matter, and may be expressed by using the term Moving Force. We must therefore say a few words on these subjects.

Measure of the Quantity of Matter.

Bodies are considered as having the same Quantity of Matter when they produce by their matter, the same mechanical effects. We have already (Art. 22.) spoken of one kind of effect produced by the matter, namely, weight; and it has been said that $2\frac{1}{2}$ cubic inches of lead will produce by its weight the same effect as $3\frac{1}{2}$ cubic inches of iron; hence $2\frac{1}{2}$

cubic inches of lead and $3\frac{1}{2}$ cubic inches of iron contain the same Quantity of Matter.

If we apply any portion of Matter to produce an effect by its weight, as for instance if we put a lump of lead into the scale of a balance, we find that it makes no difference into what shape we form the lead, if we do not add or take away any portion; and if we cut it in pieces, the effect of all the pieces is still the same as that of the whole.

Hence it appears that in such cases the Quantity of Matter depends upon the magnitude of the body only; and for any given material, the quantity of matter varies as the magnitude.

We shall shew that the velocity produced by any pressure varies as the pressure directly and as the Quantity of Matter moved inversely. Therefore another effect of Quantity of Matter is to diminish the velocity communicated by a given pressure.

This latter effect will be the same at all places and times, for it is not conceived as depending on place or time. But the weight of a body is not necessarily the same at all places; it is less at the equator than it is in our latitudes, as appears by this; that a clock pendulum, carried to the equator, goes slower than it does here. When Halley, in 1677, went to the island of St Helena to observe the stars of the southern hemisphere, he found his clock lose so much, that the screw at the bottom of the pendulum did not enable him to shorten it sufficiently. This shewed that while the Quantity of Matter which was to be moved remained the same, the pressure of gravity which moved it was diminished.

The Quantity of Matter of the same body is always the same, and we may therefore, in this manner, measure the alteration of weight at different places.

But in different bodies, *at the same place*, the Quantity of Matter is proportional to the weight, and we may therefore measure the Quantity of Matter by means of the weight.

97. The *Inertia* of a body is the effect of the Quantity of Matter in diminishing the motion impressed upon the body; when considered as a quantity it is identical with the Quantity of Matter.

The Inertia of bodies is therefore proportional to their weight at the same place; and the Inertia of the same body is always the same, though the weight should alter.

Since the velocity which is produced in a body is less as the Inertia is greater, the Inertia is sometimes spoken of as a *Resistance* to motion impressed on the body: but it appears from what is said above that the Inertia never prevents, but only diminishes the motion impressed.

The Inertia is also sometimes considered as a resistance which the body opposes to any increase or diminution of its motion; and in this sense the first Law of Motion is called the *Law of Inertia*.

The *Vis Inertiæ* is a term sometimes applied to this supposed resistance.

Measure of Moving Force.

98. The Accelerating Force is measured by the velocity produced, without taking any account of the Quantity of Matter moved. But since the circumstances of the motion depend on the Quantity of Matter as well as the velocity, the notion of Moving Force is introduced.

Moving Force is measured by the Accelerating Force multiplied into the Quantity of Matter.

The Accelerating Force and the Quantity of Matter are here both supposed to be expressed in numbers: the former is measured according to Art. 63; the latter is measured by the weight, according to Art. 96.

Cor. 1. The Moving Force varies as the Accelerating Force multiplied into the Quantity of Matter, and the Accelerating Force varies as the velocity communicated in a given time: hence the Moving Force will vary as the product of the velocity produced in a given time into the Quantity of Matter.

Cor. 2. The product of the velocity and quantity of matter is called the *Momentum:* Hence the Momentum communicated in a given time varies as the Moving Force.

The Momentum is sometimes called the *Quantity of Motion*.

The Third Law of Motion.

99. *When pressure communicates motion directly* (that is, in the direction of the pressure,) *the Moving Force is as the pressure.*

By Cor. 2, to the Definition of Moving Force, it appears that this law will be proved, if it is shewn that the momentum communicated in a given time is as the pressure which communicates it. This may be proved by experiments of the following kinds.

Suppose two bodies, each being of the weight W, to hang over a fixed pully; they will exactly balance each other. Let now a weight Q be added to one of them; the weight Q is the pressure which produces motion, and the weight moved is $2W+Q$. Hence the momentum generated in the mass $2W+Q$ at the end of one second ought to be proportional to the weight Q.

If the weights W were each 3 pounds, then, when Q is 1 pound, the velocity produced, in one second, would be $4\frac{4}{7}$ feet per second; and the momentum would be

$$(2 \times 3 + 1) \times 4\frac{4}{7} \text{ or } 32.$$

The weights W being still 3 pounds each, if Q be 2 pounds, the velocity produced in one second would be 8 feet per second, and the momentum would be $(2 \times 3 + 2) \times 8$ or 64; which is to 32 in the proportion of 2 to 1, the values of Q in the two cases.

This is an experiment in which the velocities might be made so small as to be easily measured. Thus if Q were only an ounce, the velocity produced, in one second, would be $\frac{32}{97}$ or $\frac{1}{3}$ nearly of a foot per second.

The velocities thus produced may be measured in two ways.

The weight Q may be taken off at the end of one second, and then the velocity will become uniform, and the space described in the next second will measure the velocity.

Or the space may be noticed through which Q descends in one second, which will be half the space that would have been

described by the last velocity continued uniform, because the Accelerating Force is uniform (Art. 74. Cor. 1).

Instead of one second, the spaces and velocities corresponding to any other time may be noticed; and since the Accelerating Force is uniform, the space and velocity in one second may be calculated by Art. 74.

100. In the mode of experimenting which we have just described, we have omitted to consider the Inertia of the fixed pully and the string by which the weights are connected, and also the friction and the resistance of the air. All these things however would affect the result.

Atwood's Machine is a machine which was constructed for the purpose of obtaining the results of such experiments, free from these causes of error.

The resistance of the air is got rid of, in a great degree, by making the velocities small.

The friction is much diminished by placing the axis of the fixed pully upon friction wheels.

But by this means the Inertia of the pully and wheels becomes considerable, and must not be left out of consideration.

The Inertia of the pully, wheels and string is always equivalent to a constant addition to the Inertia of the weights moved.

Let P be this addition; then when two weights W are moved by a weight Q, the Quantity of Matter moved may be considered as $2W+P+Q$, and the moving weight as Q; and with these suppositions the calculations may be made in the same manner as before.

A great number of experiments were made with such a machine; and they were all found to agree, in the most exact manner, with the results which follow from supposing the third Law of Motion, as above explained, to be true: they therefore prove this Law. An account of these experiments is given in Sect. 7 of Atwood *on Rectilinear and Rotatory Motion.*

Mr Smeaton (See *Phil. Trans.* LXVI.) also made experiments on the rotatory motion which could be produced by weights descending freely by the force of gravity. His

apparatus was not in principle much different from Atwood's Machine; and his results, though expressed in different language, agree equally well with the Laws of Motion as here stated.

101. Experiments confirmatory of the third Law of Motion may also be made by means of inclined planes; for when a body descends upon an inclined plane, the force which accelerates it has to the force of gravity a certain ratio, depending upon the inclination of the plane. It has already been stated (Art. 67.) that Galileo's experiments agreed with his theory of the motion of bodies upon the inclined plane, which theory depended upon the assumption that, for the same body, the velocity is as the pressure which produces it.

But experiments of this kind may be made more accurately by means of pendulums. If we suspend a weight at the lower end of a string of which the upper end is fixed, the force which causes this pendulum to oscillate has a certain relation to the weight of the body; and may be calculated by knowing the length of the pendulum. Thus it will appear, by the resolution of pressures, that at equal small distances from the vertical line, the pressure, or force, which urges the pendulum towards the vertical line, varies inversely as the length of the pendulum. Hence, supposing the accelerating force to be proportional to the pressure, it will appear that the time of reaching the vertical line will be as the square root of the length of the pendulum; and the time of an oscillation is the double of this time. Therefore in different pendulums the time of oscillation will be as the square root of the length of the pendulum, if the third Law of Motion be true.

Now such is found to be really the case. And this is an observation which admits of great accuracy: for the oscillations of pendulums being perpetually repeated, any deviation in the duration of an oscillation from that given by theory, is also perpetually repeated, and will thus in the course of time become sensible, however small it be.

Let there be a seconds pendulum, and another four times as long, which ought therefore to swing double seconds. If its

real time of oscillation deviates from this by $\frac{1}{1000}$ of a second, in 2000 seconds (that is in little more than half an hour) it will be wrong a whole oscillation, and will be swinging from right to left during two certain seconds, when it ought to have been swinging from left to right by the Law of Motion.

102. It appears also by Art. 92, that this third Law of Motion will be proved by proving that in the mutual action of two bodies in their direct collision, the momentum gained by one and that lost by the other in the same direction are equal.

Wren made experiments on the collision of bodies, before the Royal Society; and these experiments agreed with his theory, and therefore confirmed the third Law of Motion on which the theory depends. Experiments of the same kind were repeated by others with like success. Newton also repeated them. He hung up two bodies close to each other by strings 10 feet long, and drawing the two bodies aside to a considerable distance from each other, as 8, 12, or 16 feet, he then let them go so as to meet. He found, with the exception of inconsiderable errors, that the momenta gained and lost by the two bodies were equal in the same direction; whether the bodies were equal or unequal, elastic or inelastic. "Thus if a body *A* impinged on a body *B* at rest with a momentum 9, and after impact went on with a momentum 2, the body *B* gained the momentum 7 which *A* had lost. If the bodies met, *A* with a momentum 12 and *B* with a momentum 6, and if *A* went back with 2, *B* went back with 8, each having lost 14 in opposite directions. But if the bodies went towards the same side, *A* with a momentum 14 overtaking *B* which had a momentum of 5, and after reflexion *A* went on with 5, *B* went on with 14; *A* having lost and *B* gained 9 in the same direction. And similarly in other cases. The quantity of momentum obtained by taking the sum of conspiring or the difference of contrary motions was never changed." Newton, *Principia; Scholium to the Laws of Motion.*

Newton found the velocities of the bodies in these experiments by means of a proposition hereafter to be given, that the velocity acquired in falling down an arc of a circle (which

a pendulum does) is equal to the velocity acquired in falling down the chord.

He shewed also how allowance may be made for the velocity lost by the resistance of the air.

103. In the collision of bodies which are imperfectly elastic, the velocity with which the bodies separate after the impact is less than the velocity with which they meet before impact in the ratio of the elastic force; and this force is constant for the same kind of bodies. But the equality of the momenta gained and lost was found to obtain also in this case.

The assertion that the elastic force of bodies, as measured by the ratio of the velocity with which they separate to the velocity with which they meet, is constant for the same materials, depends upon Newton's observations. "I made the experiment" says he in the Scholium just quoted "with balls of wool, very closely pressed and tightly bound, in the following manner. First by letting go the pendulums and measuring the reflexion, I found the quantity of the elastic force; then by means of this force I determined the reflexions in other cases of collision, and the experiments answered to this determination. The balls always separated with a relative velocity which was to the relative velocity of their meeting as 5 to 9 nearly. Balls of steel separated with nearly the same velocity; those of cork with a little less; in balls of glass the proportion was 15 to 16 nearly.

And in this manner the third law, as to collision and reflexion, agrees entirely with experiment."

104. If a man in a boat pull at a rope of which the other end is fastened to a ship, both floating freely, the boat will move towards the ship, and the ship towards the boat, and the velocity of the boat will be as much greater as its quantity of matter is less; so that the momenta of the two in opposite directions will be equal.

Exactly the same will be the case if the man who pulls is in the ship instead of the boat.

If two bodies attract each other, this may be considered as equivalent to their being drawn together by an invisible cord.

They will move towards each other except they are prevented, and their opposite momenta will be equal. Thus if a magnet and a piece of iron be placed on two pieces of cork and set to float on water, they will approach each other, the smaller mass moving proportionally quicker.

105. The third Law of Motion is thus proved by experiment: having been recommended to consideration in the first place by its simplicity.

As an illustration of this simplicity it may be observed that this is the only law of the mutual action of bodies in motion, according to which the motion or rest of the center of gravity would be the same before and after the action.

Let two bodies A and B move towards each other in opposite directions with equal momenta: then the center of gravity will be at rest before impact; and the third Law of Motion being true, it will also be at rest after the impact by Art. 97. But if this Law were not true, the center of gravity, which had been at rest before the impact, would suddenly start into motion as soon as the impact took place.

Two bodies which are drawn together by a rope, or by their mutual attraction, as is supposed in Art. 104, will finally meet, and, when they have met, will be at rest, because their opposite pressures will destroy each other; therefore after their concourse their center of gravity also will be at rest. And if the third Law of Motion be true, the center of gravity will also be at rest before their concourse; but if this Law be false, the center of gravity will move with a finite velocity till the bodies meet, and will then suddenly stop.

106. Again, it follows from the third Law of Motion that the quantity of momentum of any bodies, estimated in a given direction, cannot be increased or diminished by their mutual action.

This has been shewn already in the case of the direct action of bodies, (Art. 97.) and is true in the case of their oblique action also, the resolved parts of the motions in any direction being governed by this law in the same manner as if the bodies had no other motions.

109. As a small force, acting for a long time, may produce a great momentum, so a large force, if it act for a short time, will produce only a very small effect. This is exemplified in the experiment of shooting a pistol-bullet through a sheet of paper hung loosely. Though the paper is so light, the great momentum of the bullet scarcely gives it a perceptible motion. The reason is, that the bullet acts upon the paper only during the very short time which it employs in passing through it. If the bullet have a velocity of 1000 feet a second, and the paper be one thousandth of an inch in thickness, the time of the action is only $\frac{1}{1000}$ of $\frac{1}{12}$ of $\frac{1}{1000}$ of a second, or $\frac{1}{12000000}$ of a second. If we suppose that the bullet shot into a solid mass of paper would have lost all its velocity by penetrating 1 foot, this penetration would have occupied the $\frac{1}{500}$ of a second: and hence, in the $\frac{1}{12000000}$ of a second, it would, in the same substance, lose only the $\frac{1}{24000}$ of its momentum, or the momentum corresponding to a velocity of half an inch a second; and the paper could only gain the amount of momentum which the bullet thus loses. If the paper be $\frac{1}{24}$ of the weight of the bullet, the paper will acquire a velocity of a foot a second; and therefore in $\frac{1}{12}$ of a second, which is a portion of time quite perceptible, it would only move through 1 inch. And as the resistance of the air upon a sheet of paper is very considerable, the whole velocity would be destroyed, almost before the motion could be observed.

Section III.

Motion on the Inclined Plane.

110. Prop. *The force which accelerates a body down an inclined plane, is to the accelerating force of gravity which acts on a body falling freely, as the height of the plane is to its length.*

The force which must act up an inclined plane to prevent a body W from sliding down the plane is a pressure which is to W as the height of the plane to its length (Art. 48, Cor. 1.) Hence if the body be placed upon the plane unsupported it is urged down the plane by this pressure. Therefore the moving force in this case is to the moving force when the body falls freely (or the whole weight of W) as the height of the plane to its length, (by the third Law of Motion). But the quantity of matter moved in the two cases is the same (the body W). Therefore by the definition of moving force, the accelerating force is also in the same proportion; which is the proposition to be proved. (See Art. 90.)

Cor. 1. From this it follows (by the definition of accelerating force) that the velocity produced in the same body, and *in the same time*, falling down the inclined plane and falling vertically, will be as BC to AC.

Hence, if in the time of falling down the height CB, fig. 28, a body fall down CG on the inclined plane, the velocities at G and at B are as BC to AC.

Cor. 2. Also the spaces GC, BC are as $BC : AC$ by Cor. 3, Art. 74.

Hence $GC : BC :: BC : AC$; therefore BCG, ACB, are similar triangles, and BG is perpendicular to AC.

Cor. 3. By Art. 74.

$$CA : CG :: (\text{time down } CA)^2 : (\text{time down } CG)^2$$

$$\text{that is } CA^2 : CB^2 :: (\text{time down } CA)^2 : (\text{time down } CB)^2.$$

$$\text{Hence } CA : CB :: \text{time down } CA : \text{time down } CB.$$

COR. 4. Since the velocities are as the forces,

$$\text{vel. at } B : \text{vel. at } G :: CA : CB$$

also vel. at G : vel. at A :: time in CE : time in CA; that is, :: time in CB : time in CA; and this is, by last Cor. :: CB : CA.

Hence vel. at B = vel. at A; or the velocity acquired by falling down an inclined plane is equal to the velocity acquired by falling down its vertical height.

COR. 5. Hence the velocities acquired by falling from rest down all inclined planes of the same height are equal.

It has already been shewn by experiment (Art. 89) that the velocities acquired down two circular arcs of the same height are equal. And it may be proved, upon the principles just stated, that the velocities acquired down any lines, whether curved or straight, are equal, so long as the vertical height descended through is the same.

SECTION IV.

Impact.

111. THE effect of Impact is to be calculated in each case by the principle already stated, namely, that the momenta lost and gained are equal; combined with the considerations depending upon the amount of elasticity.

The following *Example* will illustrate this.

Let an elastic ball A strike a body B at rest, which is 10 times as large as itself. If A and B were not elastic they would go on together after impact: and the momentum would be the same as A's momentum before impact. If A's original velocity be 11, the velocity after impact will be 1, in order that the momentum after impact, $(10 + 1) \times 1$, may be the same as that before impact, 1×11. Hence A loses a velocity 10 and B gains a velocity 1. But if the balls be perfectly elastic, A and B will separate with the same force with which they met:

therefore A will again lose a velocity 10 and B will gain a second velocity 1. By this means A will now move back with a velocity 9, and B move forwards with a velocity 2.

And a like mode of reasoning may be applied in other cases.

If the bodies be imperfectly elastic, it appears by experiment (Art 103.) that the relative velocity with which they separate bears a certain ratio to the velocity with which they meet, which ratio depends upon the elasticity of the material.

By combining this condition with the equality of the momenta gained and lost, the velocities of the bodies after impact may be determined in any given case from the velocties before impact.

Section V.

Simple Pendulums.

112. When a pendulum oscillates, the force which urges it towards the position in which it would be in equilibrium is not a uniform but a variable force; and therefore the propositions which have been proved concerning uniform forces cannot be immediately applied in this case. But some of the most important of the properties of oscillating pendulums may be proved by means of one or two propositions which are extensions of rules already proved for uniform forces, and are almost evident by comparison with those rules. These propositions are;

(A) In the case of uniform forces it has been proved (Arts. 72 and 73.) that the velocity acquired is as the force and the time jointly. If, in the case of two variable forces, the forces at corresponding points of the paths of the two bodies acted on, be always in a constant ratio, the motions of the two bodies will be similar; and the velocities at corresponding points will still be to each other as the forces and the times jointly.

(*B*) In the case of uniform forces it has been proved (Art. 74. Cor. 1.) that the spaces described are as the velocities and times jointly. In the case of two variable forces which at corresponding points are always in a constant ratio, the whole spaces described will also be as the velocities and times jointly.

Assuming these propositions we shall prove the following.

113. Prop. *The times of oscillation of pendulums, through equal distances from the vertical, are as the square roots of their lengths.*

Let AOo, fig. 39, be a vertical line, in which are O, o, the points of suspension of two pendulums: Pp parallel to oO meets the arcs described by the pendulums in P, p; PM, pm are horizontal. Now, by Art. 109, at the point P of the arc AP, the accelerating force in the curve is to the force on a body falling freely, as the height of the plane on which the body may be supposed to be supported is to its length: that is, as PQ to PR, AQ being horizontal: or by similar triangles PQR, PMO, as PM to PO. That is

$$\text{force on } P : \text{force of gravity} :: PM : PO;$$

and in like manner, force of gravity : force on p :: po : pm.

And as PM, pm are equal, we have

$$\text{force on } P : \text{force on } p :: po : PO.$$

Hence the forces at P, p are inversely as PO, po.

Also, if we draw any other lines parallel to Pp, the forces at the corresponding points of the arcs thus determined will also be as $po : PO$. And in this way we may divide the arcs AP, Ap into corresponding small parts.

Now since the two arcs PA, pA are described by the pendulums P, p, in virtue of the action of forces which at all corresponding points are in the same ratio, namely the ratio of po to PO, the motions in the two arcs will be similar; and the velocities produced when the bodies come to A will be as the forces and the times jointly (prop. *A*); that is, if T, t,

be the times of describing PA and pA, the velocities will be as $po \times T$ to $PO \times t$. Also, since the motions in the two arcs are similar, the spaces described will be as the velocities and the times jointly (prop. B). Therefore

$$po \times T^2 : PO \times t^2 :: AP : Ap.$$

But when the arcs AP, Ap are very small, they are very nearly equal, and may be taken as exactly equal for very small arcs; therefore $po \times T^2 = PO \times t^2$; whence

$$T^2 : t^2 :: PO : po$$

$$\text{and } T : t :: \sqrt{PO} : \sqrt{po};$$

or the times of descent to the lowest points are as the square roots of the lengths, the lengths being expressed in numbers.

The time of one oscillation or swing of the pendulum is double of the time of falling down PA; for the body P will employ an equal time in ascending on the other side of OA before it begins to return.

Cor. The time of descent down a small arc PA is very nearly the same, whatever be the magnitude of the arc. If the arc be doubled the force at each corresponding point is nearly doubled, and the whole arc is described in nearly the same time as before.

Hence the times of *any* very small oscillations of two pendulums are very nearly in the proportion of the square roots of the lengths.

It is found that the length of a pendulum which oscillates once a second at London is 39,1386 inches. Hence we can solve various problems of the following kind. It is to be observed that in such problems the oscillations are always supposed to be very small.

Ex. 1. To find the time of oscillation of a pendulum 20 feet long: 20 feet = 240 inches: hence

$$\sqrt{39{,}1386} : \sqrt{240} :: 1 : 2{,}5 \text{ nearly.}$$

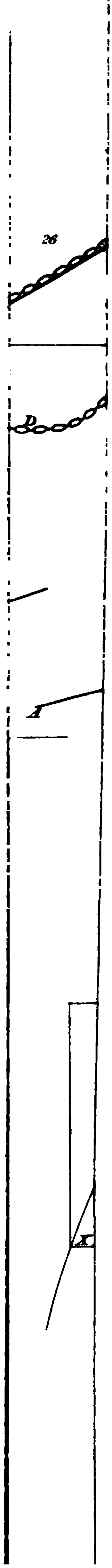
The time of oscillation is $2\frac{1}{2}$ seconds.

Ex. 2. A ball, suspended by a string from the roof of a high building, oscillates 10 times a minute: what is the height of the roof?

The time of oscillation is here 6 seconds. Hence

$$1^2 : 6^2 :: 39{,}1386 \text{ inches} : 117{,}4158 \text{ feet},$$

which is the height.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Alexander Hewett

ANALYTICAL STATICS.

A SUPPLEMENT

TO THE

FOURTH EDITION OF AN ELEMENTARY TREATISE

ON

MECHANICS.

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BY W. WHEWELL, M.A. F.R.S.

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M.DCCC.XXXIII.

PREFACE.

HAVING thought it advisable, in the fourth edition of my "Elementary Treatise on Statics," to separate from the absolutely elementary portion, those parts which assume the Student to possess a knowledge of Analytical Geometry and of the Differential Calculus, I have ventured to affix to this separate publication the title of Analytical Statics. But this title is to be understood rather as indicating the nature of the subject, than as promising a complete Treatise upon it. Such a Treatise would imply an extension of the plan of the former work for which I was not prepared when a new edition was called for. I have, however, inserted a few of the deficient propositions which seem most important in such a work; for instance, an independent proof of the Composition of Forces acting at a point, (Chap. I.) and a proof of the general principle of Virtual Velocities (Art. 20—22). In addition to these, there are other propositions, remarkable for their Analytical generality or beauty, which might properly form parts of a Treatise on Statics: for instance;—the Theory of Moments;—and certain propositions founded upon the principle of Virtual Velocities. For these subjects I may refer to Mr Poisson's Treatise, Articles 271 to 284, and 346 to 349, of the Second Edition.

I have, for reasons already stated, been desirous of introducing, as far as can conveniently be done, propositions which have a bearing upon practical applications of mechanical knowledge; and have, with this view, bor-

rowed from the excellent Memoirs of Mr Davies Gilbert and Mr Hodgkinson, on Suspension Bridges, some of the most important portions. I have also introduced a Chapter upon the Strength of Materials with regard to Fracture. It is very obviously desirable that this subject should enter into our Treatises of Mechanics; but the difficulty of effecting this may be supposed to be considerable, when we perceive that the theory of Galileo, which assumes that materials are absolutely incapable of being either condensed or broken by compression, has held its place in some of our most received Treatises up to the present time; notwithstanding its complete repugnance both to our general conceptions of the structure of materials, and to the results of observation. The researches of Mr Barlow and others, and still more recently the well devised experiments and clear views of Mr Hodgkinson, have, it may be hoped, done much to put us in possession of a theory of this subject, consistent with itself and with facts. In this hope I have endeavoured to bring the subject before the Student of Mechanics, following principally the investigations of Mr Hodgkinson, as contained in the Transactions of the Manchester Society. Though discrepancies and difficulties may still exist with regard to this matter, they will probably disappear in the course of further researches, if the fundamental principles are rightly established.

I have omitted the investigations concerning the Forms of Bridges on various hypotheses, and the discussion of the Species of the Elastic Curve, which made part of the former Editions. These portions of the work are not suited to the Mathematical Student as parts of a Course of Mechanics, and occupied too much space for mere examples.

CONTENTS.

ERRATA.

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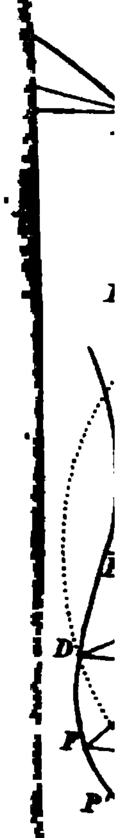
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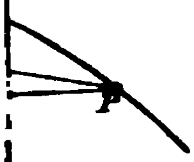

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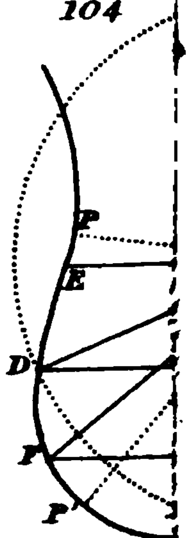

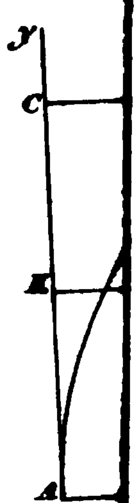
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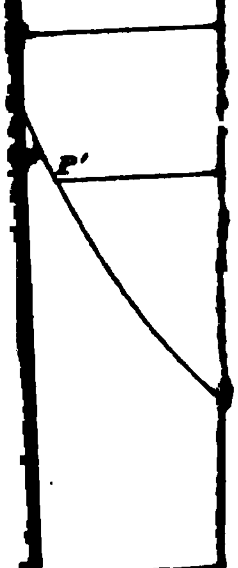
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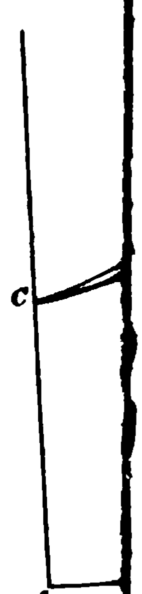
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ANALYTICAL STATICS.

CHAP. I.

RESOLUTION OF A FORCE INTO RECTANGULAR COMPONENTS.

1. Statics is that part of the science of Mechanics which treats of forces employed in producing equilibrium.

The forces treated of in this part of Mechanics are *pressures*. They are measured by the number of units of pressure to which they are equivalent; each unit of pressure being supposed capable of producing an equal effect in maintaining equilibrium.

When two or more forces act at the same point in any directions, they produce an effect which may be produced by a single force acting at the same point. The two forces are said to be *compounded* into the single force; the two forces are called the *components;* the single force is called the *resultant*. If the single force be the one first considered, it is said to be *resolved* into the two component forces.

When two forces act in the same direction, their resultant is their sum; when they act in opposite directions, their resultant is their difference, and is in the direction of the greater.

If a point be acted upon by any two forces, and also by a force equal and opposite to their resultant, it will be kept in equilibrium: for this is equivalent to its being acted on by the resultant and by a force equal and opposite to the resultant.

2. The relations of forces which keep each other in equilibrium may be deduced, beginning either with the con-

sideration of forces acting at a point, or of forces acting on a lever. Each case may be deduced from the other. In the Elementary Treatise, to which the present volume is a Supplement, the latter plan was followed. It was there shewn (Art. 27), that if two forces p, q, act at any angle, their resultant r is represented, in magnitude and direction, by the diagonal of a parallelogram of which the sides similarly represent p, q. It hence follows, that if p, q act at right angles to each other, and if θ be the angle which r makes with p, we have

$$p = r\cos\theta, \quad q = r\sin\theta, \quad \tan\theta = \frac{q}{p}, \quad r = \sqrt{(p^2 + q^2)}.$$

By means of these expressions we may treat all the problems of Statics analytically; and this we shall proceed to do for some of the most important.

But, in order to make the analytical mode of treating the subject more complete, we shall first prove the above expressions independently, beginning with the consideration of forces acting at a point.

3. Prop. *If a force* r *be resolved into two forces* p, q, *at right angles to each other, and if θ be the angle between* r *and* p, *the ratio* $\frac{\mathrm{p}}{\mathrm{r}}$ *is the same so long as θ is the same.*

Let n equal forces r', r', &c. act in the direction of r; and let p', q' be the components of each in the directions of p, q; the components of each of the equal forces r', r', &c. will be equal. Hence, nr' will be the force in the direction of r, and np', nq' the components, whatever n may be. And the angle θ is the angle between r' and p', and therefore between r and p. Therefore θ is the same so long as r and p are represented by nr' and np'; that is, so long as r and p are in the ratio of r' and p'.

Hence, θ is the same so long as $\frac{p}{r}$ is the same, and one of these quantities varies only when the other does. Therefore also $\frac{p}{r}$ is the same so long as θ is the same.

COR. 1. We may express this dependence by saying, that $\frac{p}{r}$ is a *function* of θ; or in symbols

$$\frac{p}{r} = \phi(\theta),$$

$\phi(\theta)$ representing a function of θ hereafter to be determined.

COR. 2. It is obvious that q will depend on $\frac{\pi}{2} - \theta$, in the same manner in which p depends on θ; hence

$$\frac{q}{r} = \phi\left(\frac{\pi}{2} - \theta\right).$$

We shall in what follows suppose p, q to be the rectangular components of r, and θ the angle between r and p; and shall express the dependence of p on r and θ in the manner just stated; whence we have $p = r\phi(\theta)$.

4. PROP. *If two forces, each equal to* r, *act at an angle* 2θ, *and produce a resultant* s, *we shall have*

$$s = 2r\phi(\theta).$$

Let AR, AR', fig. 91, be the directions in which the equal forces r, r' act; so that RAR' is 2θ. Bisect the angle RAR' by the line AS, and draw QAQ' perpendicular to AS.

Let the force r, which acts in AR, be resolved into p acting in the direction AS, and q acting in the direction AQ. Then by the last Article, since $RAS = \theta$,

$$p = r\phi(\theta), \quad q = r\phi\left(\frac{\pi}{2} - \theta\right).$$

In like manner, the force r' equal to r, which acts in AR' may be resolved into p' acting in AS and q' acting in AQ'; and, as before,

$$p' = r\phi(\theta), \quad q' = r\phi\left(\frac{\pi}{2} - \theta\right).$$

The two forces, q, q', are equal and in opposite directions, and therefore destroy each other; and therefore the resultant

of the forces r, r', is the resultant of p and p', which, since they are in the same direction, is the sum of p and p', or $2r\phi(\theta)$. Therefore the resultant $s = 2r\phi(\theta)$.

5. PROP. *If, for any given angle* 2θ, *we have* $\phi(2\theta) = \cos. 2\theta$, *we shall also have* $\phi(\theta) = \cos. \theta$.

Let, in fig. 92, two equal forces q, q' act at an angle QAQ' which is 4θ; the resultant will be s, in the direction AS which bisects PAP', and we shall have $s = 2q\phi(2\theta)$.

Suppose that besides the two forces q, q', in AQ, AQ', two forces p, p', equal to these, act in AP. Then the resultant of the four forces p, p', q, q' will be $2p + 2q\phi(2\theta)$.

But the two equal forces p in AP and q in AQ are equivalent to a force r in AR which bisects QAS. And in like manner the two forces p in AP and q in AQ' are equivalent to a force r in AR'. And by last Art. we have in each case

$$r = 2p\phi(\theta).$$

Again, the two forces r in AR and r in AR' are equivalent to a force $2r$, $\phi(\theta)$ by the same Article; that is, putting for r its value, the two forces r, r', or the four forces p, p', q, q', are equivalent to $4p\,\{\phi(\theta)\}^2$.

Hence we have, putting p for q

$$4p\,\{\phi(\theta)\}^2 = 2p + 2p\phi(2\theta);$$

$$\phi(\theta) = \sqrt{\frac{1 + \phi(2\theta)}{2}}.$$

By comparing this with the trigonometrical formula

$$\cos. \theta = \sqrt{\frac{1 + \cos. 2\theta}{2}},$$

it appears that if $\phi(2\theta)$ is $\cos. 2\theta$, $\phi\theta$ will be $\cos. \theta$.

COR. It appears also that if $\phi(\theta)$ is $\cos. \theta$, $\phi(2\theta)$ will be $\cos. 2\theta$.

6. Prop. *For all angles which can be obtained by the continual bisection of an angle of* 60 *degrees,* $\phi(\theta)$ *is* cos. θ.

Let two equal forces q, q', act in AQ, AQ', fig. 92, at an angle of 120°, so that θ is 60°. Their resultant, s, in the direction AP, will bisect the angle QAQ', whence PAQ will be 60°; and if PA be produced to O, QAO will be 120°. If a force s' equal to s, the resultant of q, q', be applied at A in the direction AO, the three q, q', s', will keep the point A in equilibrium. But in this case, since the three angles QAQ', QAO, $Q'AO$ are equal, the three forces q, q', s' must be equal: for each may be considered as the resultant of the other two, and each pair act at the same angle. Therefore $s' = q$, and $s = q$.

Now by the last Article $s = 2q\phi(\theta)$; whence we have $q = 2q\phi(\theta)$ and $\phi(\theta) = \frac{1}{2}$, when θ is 60°. And cos. $60° = \frac{1}{2}$; whence it appears that in this case $\phi(\theta)$ is cos. (θ).

Hence it follows, by the last Article, that $\phi(\theta)$ is cos. θ, whence θ is 30°. Hence again $\phi(\theta)$ is cos. θ, when θ is 15°; and so on, to any arc which can be obtained by the continual bisection of 60°.

Cor. By continual bisection we may make the angle θ smaller than any assigned angle, and the proposition will still be true.

7. Prop. *If* $\phi(\theta)$ *be* cos. θ, *and* $\phi(\epsilon)$ *be* cos. ϵ, *and* $\phi(\theta-\epsilon)$ *be* cos. $(\theta-\epsilon)$; *then also* $\phi(\theta+\epsilon)$ *is* cos. $(\theta+\epsilon)$.

Let two equal forces r, r', in AR, AR', fig. 93, make an angle 2θ; and let s in AS be their resultant. Let r, which acts in AR, be the resultant of two equal forces p, q, acting in AP, AQ, at an angle 2ϵ. Therefore RAP will be ϵ, and $r = 2p\,\phi(\epsilon)$. Also SAR will be θ, and $s = 2r\,\phi(\theta)$

$$= 4p\,\phi(\epsilon)\,.\,\phi(\theta).$$

The force s is the resultant of the two forces p, p' acting in AP, AP', and the two q, q' acting in AQ, AQ'. And the two equal forces p, p', will have a resultant in AS, which

by Art. 4. will be $2p\,\phi(\theta-\epsilon)$, since PAS is $\theta-\epsilon$. Also the two q, q' will have a resultant $2q\,\phi(\theta+\epsilon)$ in AS, since QAS is $\theta+\epsilon$. Hence s, the resultant of the force p, p', q, q', will be $2p\,\phi(\theta-\epsilon)+2q\,\phi(\theta+\epsilon)$; and since q is equal to p, equating this with the former value of s, we have

$$2p\phi(\theta-\epsilon)+2p\phi(\theta+\epsilon)=4\phi(\epsilon)\,.\,\phi(\theta).$$

$$\text{Hence } \phi(\theta+\epsilon)=2\phi(\epsilon)\,.\,\phi(\theta)-\phi(\theta-\epsilon).$$

Now we have by trigonometry,

$$\cos.(\theta+\epsilon)=2\cos.\epsilon\,.\cos.\theta-\cos.(\theta-\epsilon).$$

If therefore

$\phi(\theta)$ be $\cos.\theta$, $\phi(\epsilon)$ be $\cos.\epsilon$, and $\phi(\theta-\epsilon)$ be $\cos.(\theta-\epsilon)$,

we shall have also $\phi(\theta+\epsilon)=\cos.(\theta+\epsilon)$.

8. Prop. *For all values of* θ, $\phi(\theta)$ *is* $\cos.\theta$.

Whatever be the value of θ, we may, by the perpetual bisection of an angle of 60^0, obtain an angle which either measures θ, or measures it with a remainder less than any assigned angle, since we may by the perpetual bisection of an angle of 60^0 obtain an angle less than any assigned angle. We will therefore suppose $\theta=n\delta$, when δ is an arc obtained by the perpetual bisection of an angle of 60^0, and n is a a whole number.

It is proved, Art. 6, that $\phi(\delta)$ is $\cos.\delta$. It hence follows by Cor. to Art. 5, that $\phi(2\delta)$ is $\cos.2\delta$.

Also since this is true for 2δ, δ, and $2\delta-\delta$, it is true, by Art. 7, for $2\delta+\delta$ or 3δ.

Again since it is true for 3δ, δ, and $3\delta-\delta$ or 2δ, it is true for $3\delta+\delta$, or 4δ.

Again since it is true for 4δ, δ, and $4\delta-\delta$ or 3δ, it is true for $4\delta+\delta$, or 5δ.

And in this manner it may be proved for $n\delta$, when n is any whole number.

Therefore, by what has been said, it is true for θ.

Hence if p, q be the rectangular components of a force r, of which p makes with r an angle θ,

$$p = r\cos.\theta.$$

Cor. 1. Hence also $q = r\cos.\left(\frac{\pi}{2} - \theta\right) = r\sin.\theta.$

Cor. 2. Hence $\tan.\theta = \frac{r\sin.\theta}{r\cos.\theta} = \frac{p}{q}.$

Cor. 3. Also $r^2 = r^2\cos^2.\theta + r^2\sin.^2\theta = p^2 + q^2.$

CHAP. II.

THE CONDITIONS OF EQUILIBRIUM OF A POINT.

9. In this and the following Chapter we shall express the conditions which are requisite that a point or a body may be in equilibrium, by means of equations among the symbols which the forces and their positions introduce; and we shall thus obtain the means of reducing to the solution of equations, all problems whatever relative to equilibrium.

Prop. *To find the resultant of two forces acting at a point, as AP, AQ, fig.* 94.

If we suppose a line, as Ax, the position of which is known, to pass through the point A, we may determine the positions, both of the components, and of the resultant, by the angles which they make with this line.

Let p, q, be the forces in AP, AQ; α, β, the angles which they make with Ax. If p be resolved into two forces,

one in the direction Ax, and the other in the direction Ay perpendicular to Ax; it has been shewn, in the preceding Chapter, that these resolved parts will be $p \cos. \alpha$, $p \sin. \alpha$. In the same manner q is equivalent to forces $q \cos. \beta$ in the direction Ax, and $q \sin. \beta$ in the direction Ay. Hence the forces p, q are equivalent to

$$p \cos. \alpha,\ q \cos. \beta \text{ in } Ax,$$
$$p \sin. \alpha,\ q \sin. \beta \text{ in } Ay;$$

and the resultant of p and q will be the resultant of these four forces. If we put

$$p \cos. \alpha + q \cos. \beta = X,$$
$$p \sin. \alpha + q \sin. \beta = Y;$$

and if r be the resultant of p and q, and θ the angle which it makes with Ax, we have by Art. 8, Cor. 2, 3,

$$r = \sqrt{(X^2 + Y^2)},\ \tan. \theta = \frac{Y}{X}.$$

whence the magnitude and position of the resultant are known.

Cor. 1. By putting the values of X and Y in the expression for r, we find

$$r = \sqrt{\begin{Bmatrix} p^2 \cos.^2 \alpha + 2pq \cos. \alpha . \cos. \beta + q^2 \cos.^2 \beta \\ + p^2 \sin.^2 \alpha + 2pq \sin. \alpha . \sin. \beta + q^2 \sin.^2 \beta \end{Bmatrix}},$$

and since $\cos.^2 \alpha + \sin.^2 \alpha = 1$,

and $\cos. \alpha . \cos. \beta + \sin. \alpha . \sin. \beta = \cos. (\alpha - \beta)$,

$$r = \sqrt{\{p^2 + 2pq \cos. (\alpha - \beta) + q^2\}}.$$

Cor. 2. This agrees with the result obtained in Chap. II. of the Elementary Treatise; for if AP, AQ represent the forces p, q, and if AR be found by completing the parallelogram $APRQ$, we shall have

$$AR^2 = AP^2 + PR^2 + 2AP . PR . \cos. RPE,$$
$$\text{or } = p^2 + q^2 + 2pq \cos. (\alpha - \beta),$$
$$\text{because } RPE = QAP = PAx - QAx.$$

COR. 3. If we call the angles PAR and QAR, ϕ and ψ respectively, we shall have, by Trigonometry,

$$\frac{\sin. PAR}{\sin. APR} = \frac{PR}{AR}, \text{ or}$$

$$\frac{\sin. PAR}{\sin. EPR} = \frac{A}{AR};$$

$$\therefore \frac{\sin. \phi}{\sin. (\alpha - \beta)} = \frac{q}{r},$$

$$\sin. \phi = \frac{p \sin. (\alpha - \beta)}{r} = \frac{q \sin. (\alpha - \beta)}{\sqrt{\{p^2 + 2pq \cos. (\alpha - \beta) + q^2\}}}.$$

Similarly

$$\sin. \psi = \frac{p \sin. (\alpha - \beta)}{r} = \frac{p \sin. (\alpha - \beta)}{\sqrt{\{p^2 + 2pq \cos. (\alpha - \beta) + q^2\}}}.$$

10. PROP. *To find the resultant of any number of forces* $p_1, p_2, p_3 \ldots p_n$ *in the same plane; their directions making with the line* Ax, *angles* $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ *respectively.*

As in the last Article, Ay being perpendicular to Ax, the forces may be shewn to be equivalent to

$p_1 \cos. \alpha_1, p_2 \cos. \alpha_2, p_3 \cos. \alpha_3 \ldots p_n \cos. \alpha_n$ in the direction Ax,
$p_1 \sin. \alpha_1, p_2 \sin. \alpha_2', p_3 \sin. \alpha_3 \ldots p_n \sin. \alpha_n$ in the direction Ay.

Hence, if r be the resultant, and θ the angle which it makes with Ax, r and θ will be given by the equations

$$p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 \ldots + p_n \cos. \alpha_n = X,$$
$$p_1 \sin. \alpha_1 + p_2 \sin. \alpha_2 + p_3 \sin. \alpha_3 \ldots + p_n \sin. \alpha_n = Y,$$

$$r = \sqrt{(X^2 + Y^2)}; \ \tan. \theta = \frac{Y}{X}.$$

We have considered the forces as lying within the angle yAx and *pulling* the body. In this case the resolved parts will be in the directions Ax and Ay; but if one of the forces act in the direction AP', fig. 95, situated in the angle yAx',

the resolved part AM' will act in the direction xA, and the corresponding term is the sum $p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 +$ &c. should be negative. And if $p' \cos. \alpha'$ be this term, it will be negative, because $\alpha' = P'Ax$, and $p' \cos. \alpha' = p' \cos. P'Ax = -p' \cos. P'Ax'$, which is a negative quantity. In this case $p' \sin. \alpha'$ will be positive, which agrees with the direction of the resolved force $M'P'$.

In the same manner, if a force p'' act in the direction AP'', in the quadrant $y'Ax$, the term $p'' \sin. \alpha''$ will be negative, and $p'' \cos. \alpha''$ will be positive.

And if a force p''' act in the direction AP''' in the quadrant $y'Ax'$, the terms $p''' \cos. \alpha'''$, $p''' \sin. \alpha'''$, will both be negative. And these changes of sign agree with the changes of direction of the resolved parts.

And if the force, instead of being a *pulling* force in the direction AP, be a *pushing* force in the direction PA, we must make p negative; and the resolved parts $p \cos. \alpha$ and $p \sin. \alpha$ will both be negative. In the same manner if the force in $P'A$ be a pushing force, we must make p' negative. And similarly in the other quadrants.

11. PROP. *To find the resultant of forces whose directions are not all in the same plane.*

We have in the preceding case resolved forces in the directions of two lines at right angles to each other. In this case we shall resolve them in the directions of three lines, each at right angles to the other two. The nature of space admits of three such lines, or *axes*, and no more. Let xAy, fig. 96, be conceived to be a horizontal plane, in which Ax and Ay are at right angles; and let Az be vertical. Then Ax, Ay, Az are all at right angles to each other; and the planes xAy, xAz, yAz are also at right angles to each other. For (Euc. XI. Def. 6.), yAx measures the inclination of zAy, zAx. And similarly of the others.

Let P be any point in space; and through P let three planes be drawn, $PmOn$, $PoNm$, $PoMn$, parallel respectively

to xAy, xAz, yAz. Hence Mm will be a rectangular parallelopiped; and therefore the plane nMo is perpendicular to AMo, AMn. Therefore AM is perpendicular to the plane nMo (Euc. XI. 19.), and therefore to the line PM (Euc. XI. 4.).

If AP represent any force acting at A, AP may be resolved into forces represented by AM, MP. Also MP may be resolved into Mo, oP; and hence the force AP is equivalent to AM, Mo, oP; or to AM, AN, AO.

Since PM is perpendicular to AM, $AM = AP \,.\, \cos. PAx$. And similarly $AN = AP \,.\, \cos. PAy$ and $AO = AP \,.\, \cos. PAz$. Hence if p be the force AP, and α, β, γ, the angles which it makes with Ax, Ay, Az, the force will be equivalent to three forces

$$p \cos. \alpha \text{ in } Ax, \quad p \cos. \beta \text{ in } Ay, \quad p \cos. \gamma \text{ in } Az^*.$$

Hence if we have forces $p_1, p_2, p_3, \ldots p_n$, acting at a point A

$$\begin{aligned} &\text{making with } Ax_1 \text{ angles } \alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n; \\ &\text{with } Ay_1 \text{ angles } \beta_1, \beta_2, \beta_3, \ldots \beta_n; \\ &\text{with } Az_1 \text{ angles } \gamma_1, \gamma_2, \gamma_3, \ldots \gamma_n; \end{aligned}$$

and if we make

$$p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 \ldots + p_n \cos. \alpha_n = X;$$

$$p_1 \cos. \beta_1 + p_2 \cos. \beta_2 + p_3 \cos. \beta_3 \ldots + p_n \cos. \beta_n = Y;$$

$$p_1 \cos. \gamma_1 + p_2 \cos. \gamma_2 + p_3 \cos. \gamma_3 \ldots + p_n \cos. \gamma_n = Z;$$

the forces will be equivalent to X in Ax, Y in Ay, and Z in Az.

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* *Two* of the angles α, β, γ are sufficient to determine the position of the line AP, for they are connected by the equation

$$\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = 1;$$

so that two of them being known, the third may be found.

This appears thus;

$$\begin{aligned} AP^2 &= AM^2 + MP^2 = AM^2 + Mo + oP^2 \\ &= AM^2 + AN^2 + PO^2 \\ &= AP^2 \cos.^2 \alpha + AP^2 \cos.^2 \beta + AP^2 \cos.^2 \gamma; \\ \therefore 1 &= \cos. \alpha^2 + \cos.^2 \beta + \cos.^2 \gamma. \end{aligned}$$

If R be the resultant, and θ, η, ζ the angles which it makes with Ax, Ay, Az respectively, we shall have

$$R = \sqrt{(X^2 + Y^2 + Z^2)},$$

$$\cos.\theta = \frac{X}{R}, \cos.\eta = \frac{Y}{R}, \cos.\zeta = \frac{Z}{R}.$$

For if AM, AN, AO now represent X, Y, Z, AP will represent R; and $AP^2 = AM^2 + AN^2 + AO^2$ (see note last page).

Also $AM = AP \cos. PAM$, &c.

One of the three last equations is superfluous, as was observed before.

As in the last Article, the resolved forces may become negative when the angles α_1, β_1, γ_1, &c. pass beyond the first quadrant. Also the forces are negative when they push instead of pulling.

12. Prop. *When a point is acted upon by any forces, to find the conditions of equilibrium.*

In order that there may be an equilibrium, the resultant of all the forces must be 0. And in order that this may be the case it is evident that we must have, in Art. 10, $X = 0$, $Y = 0$; and, in Art. 11, $X = 0$, $Y = 0$, $Z = 0$. Hence we have for the conditions of equilibrium in the former case,

$$p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 + \ldots = 0;$$

$$p_1 \sin. \alpha_1 + p_2 \sin. \alpha_2 + p_3 \sin. \alpha_3 + \ldots = 0.$$

And in the latter case

$$p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 + \ldots = 0;$$

$$p_1 \cos. \beta_1 + p_2 \cos. \beta_2 + p_3 \cos. \beta_3 + \ldots = 0;$$

$$p_1 \cos. \gamma_1 + p_2 \cos. \gamma_2 + p_3 \cos. \gamma_3 + \ldots = 0.$$

The conditions of the equilibrium of any number of points may be deduced from the conditions belonging to one point. In the state of equilibrium, each point, by

means of the rods, strings, &c. which connect it with the other points, exerts and suffers a certain pressure. And this pressure may be introduced as one of the forces at each point, and then eliminated by considering that it is equal at each two points so connected.

13. PROP. *When a body is supported upon a curve (the curve being in a vertical plane); to find the conditions of equilibrium.*

Let AM, MP, fig. 97, be the vertical abscissa and the horizontal ordinate of the curve; and let $AM = x$, $MP = y$, $BP = s$. Let the forces which act on the body be resolved in the directions parallel to x and to y, and let the resolved parts thus obtained be called X and Y: X and Y being considered positive when they tend to increase x and y. Also let R be the re-action of the curve in the direction of the normal, or what is the same thing, the pressure of the body on the curve. Then, in order to obtain the conditions of equilibrium, resolve R in the directions parallel to x and to y;

$\therefore$ resolved part of R in direction $PX = R \cos. RPX$

$$= R \sin. XPT = R \, . \frac{dy}{ds},$$

resolved part of R in direction $PY = R \cos. RPY$

$$= -R \cos. RPM = -R \, . \frac{dx}{ds}.$$

Hence, by Art. 12, the equilibrium will subsist if

$$X + R\frac{dy}{ds} = 0;$$

$$Y - R\frac{dx}{ds} = 0.$$

The first of these equations is equivalent to

$$X + R\frac{dy}{dx}\frac{dx}{ds} = 0$$

and if we multiply the second by $\dfrac{dy}{dx}$ and add it to this, we have

$$X + Y\frac{dy}{dx} = 0,$$

which is the equation of equilibrium. If we know the curve, that is, the relation between x and y, this equation will give us the relation between X and Y; and if we know this also, the equation will enable us to find the actual values of x and y, or the point when the body will be supported. This will be illustrated by the problems which follow.

If the weight P, instead of resting upon a material surface BP, fig. 97, be suspended by a string KP which confines it to the curve BP, the conditions of equilibrium will be the same as before. The re-action which was before supplied by the resistance of the surface is now produced by the tension of the string. This re-action will as before be perpendicular to the curve: it will also manifestly be in the direction of the string, and this agrees with what is collected from the way in which the curve is described; for when a curve is traced out by one end of a string of which the other is fixed, the string will at every point be perpendicular to the curve Hence the formulæ which we are about to give for the former case apply also to this.

14. Prop. *A body is supported upon a curve by a weight acting over a fixed pully K, fig.* 98; *to find the conditions of equilibrium.*

Take the vertical line KM, passing through the pully, for the line on which x is measured downwards.

Let $KM = x$, $MP = y$, $KP = r = (x^2 + y^2)^{\frac{1}{2}}$; and if the weight which acts by means of KP be $= q$, the parts which act parallel to MK and PM are

$$q\frac{x}{r}, \text{ and } q\frac{y}{r};$$

$$\text{hence } X = p - q\frac{x}{r}; \quad Y = -q\frac{y}{r};$$

hence the equation of Art. 13, namely, $X + Y\frac{dy}{dx} = 0$, becomes

$$p - q\frac{x}{r} - q\frac{ydy}{rdx} = 0,$$

$$\text{or } p - q\left(\frac{x}{r} + \frac{ydy}{rdx}\right) = 0;$$

$$\text{but } x^2 + y^2 = r^2; \ \therefore \ x + \frac{ydy}{dx} = \frac{rdr}{dx},$$

$$\text{and } \frac{x}{r} + \frac{ydy}{rdx} = \frac{dr}{dx};$$

$$\therefore \ p - q\frac{dr}{dx} = 0,$$

and this, combined with the relation between x and r, which is given by the nature of the curve, gives the position of equilibrium.

15. PROB. I. *Let AP, fig. 98, be a hyperbola with its axis vertical, on which a given weight P is supported by another given weight Q by means of a string passing over a pully at the center; to find the position of equilibrium.*

Let as before KM, MP, AP, be x, y, r; the semi-axes of the hyperbola a and b: the given weights p and q.

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2);$$

$$\therefore \ r = (x^2 + y^2)^{\frac{1}{2}} = \left(x^2 + \frac{b^2x^2}{a^2} - b^2\right)$$

$$= \left(\frac{a^2 + b^2}{a^2}x^2 - b^2\right)^{\frac{1}{2}}$$

$$= (e^2x^2 - b^2)^{\frac{1}{2}}:$$

making $e = \frac{(a^2 + b^2)^{\frac{1}{2}}}{a}$, which is called the *eccentricity* of the hyperbola;

$$\therefore \frac{dr}{dx} = \frac{e^2 x}{(e^2 x^2 - b^2)^{\frac{1}{2}}}; \text{ hence } p - q\frac{dr}{dx} = 0 \text{ becomes}$$

$$p - q\frac{e^2 x}{(e^2 x^2 - b^2)^{\frac{1}{2}}} = 0;$$

$$\therefore p^2 (e^2 x^2 - b^2) = q^2 e^4 x^2;$$

$$\therefore x^2 = \frac{p^2 b^2}{e^2 (p^2 - q^2 e^2)};$$

$$\therefore x = \frac{pb}{e (p^2 - q^2 e^2)^{\frac{1}{2}}}; \text{ and hence we may find } y, r.$$

Cor. 1. If $q^2 e^2 > p^2$, or $qe > p$, the equilibrium is impossible.

Cor. 2. If $qe = p$, $x = \infty$: the body would in this case be supported upon the asymptote.

16. Prob. II. *It is required to find a curve such that a given weight* = q *hanging over the pully may balance another given weight* = p *at every point of it.*

We must have at every point

$$p - q\frac{dr}{dx} = 0;$$

hence, integrating with respect to x,

$$px - qr + c = 0;$$

$$\therefore px + c = qr = q (x^2 + y^2)^{\frac{1}{2}};$$

$$\therefore p^2 x^2 + 2pcx + c^2 = q^2 x^2 + q^2 y^2;$$

$$\therefore y^2 = \frac{p^2 - q^2}{q^2} x^2 + \frac{2pc}{q^2} x + \frac{c^2}{q^2}.$$

Let $x + \dfrac{pc}{p^2 - q^2} = t$;

$$\therefore (p^2 - q^2) x^2 + 2pcx + \frac{p^2 c^2}{p^2 - q^2} = (p^2 - q^2) t^2,$$

$$y^2 = \frac{1}{q^2}\left\{(p^2 - q^2)\,t^2 - \frac{p^2c^2}{p^2 - q^2} + c^2\right\};$$

$$= \frac{p^2 - q^2}{q^2}\left(t^2 - \frac{q^2c^2}{(p^2 - q^2)^2}\right).$$

But if t, y, be the abscissa and ordinate of a hyperbola in which the semi-axes are a, b,

$$y^2 = \frac{b^2}{a^2}(t^2 - a^2);$$ which agrees with our equation, if

$$\frac{p^2 - q^2}{q^2} = \frac{b^2}{a^2}, \text{ and } \frac{q^2c^2}{(p^2 - q^2)^2} = a^2:$$

$$\text{hence } \frac{c^2}{p^2 - q^2} = b^2.$$

Hence the curve required is a hyperbola in which $KM = x$, and $CM = t$; fig. 99; and in which the semi-axes are

$$a = \frac{qc}{(p^2 - q^2)}, \text{ and } b = \frac{c}{(p^2 - q^2)^{\frac{1}{2}}};$$

$$CK \text{ is } = \frac{pc}{p^2 - q^2} = \frac{pa}{q}.$$

$$(a^2 + b^2)^{\frac{1}{2}} = \left(\frac{q^2c^2}{(p^2 - q^2)^2} + \frac{c^2}{(p^2 - q^2)}\right)^{\frac{1}{2}} = \frac{pc}{p^2 - q^2} = CK;$$

$\therefore$ K is the focus.

If we call AK, k, we have

$$k = CK - CA = \frac{pc}{p^2 - q^2} - \frac{qc}{p^2 - q^2} = \frac{c}{p + q};$$

$$\therefore\ c = (p + q)\,k,$$

and putting this value for c, the semi-axes become

$$a = \frac{q}{p - q}k, \text{ and } b = \left(\frac{p + q}{p - q}\right)^{\frac{1}{2}} k.$$

17. PROP. *Two given weights P, P', connected by a string of given length* (= b) *passing over a given pully K, fig.* 100, *are supported on two curves, which are in the same vertical plane as the pully. Having given one curve, to find the other so that the weights may balance in every position.*

Let the weights be p, p', and the tension of the string q. And let x, x', r, r', be the values of the abscissæ KM, KM', and of KP, KP'. Then since q must be equal to a weight which, hanging freely, would support either P or P', we have by the last Problem,

$$p - q\frac{dr}{dx} = 0, \text{ and } p' - q\frac{dr'}{dx'} = 0.$$

The second equation multiplied by $\dfrac{dx'}{dx}$ gives

$$p'\frac{dx'}{dx} - q\frac{dr'}{dx} = 0;$$

$$\text{and since } r + r' = b,\ \frac{dr}{dx} + \frac{dr'}{dx} = 0;$$

whence, if we add the first and the third equations, we have

$$p + p'\frac{dx'}{dx} = 0; \text{ and integrating in } x;$$

$\therefore\ px + p'x' = c$. This equation, along with the one $r + r' = b$, enables us to find x' and r' in terms of x and r: and as we know the nature of the curve $A'P'$, we have the relation between r' and x', which we may represent thus, $r' = f(x')$, $f(x')$ representing a known function of x'; and by substituting the values of x' and r' we have a relation between x and r, which determines the curve required.

18. PROB. III. *As an example, suppose the given curve $A'P'$, fig.* 101, *to be a circle, and CK a vertical line through its center:* and let $KC = k$, $A'C$, the radius of the circle, $= a$; then,

$$KP'^2 = KC^2 + CP'^2 - 2KC\,.\,CM', \text{ or}$$

$$r'^2 = k^2 + a^2 - 2k(k - x') = a^2 - k^2 + 2kx',$$

$$\text{or since } r' = b - r, \text{ and } x' = \frac{c - px}{p'}, \text{ by last Article;}$$

$$\therefore\ (b - r)^2 = a^2 - k^2 + \frac{2k}{p'}(c - px),$$

$$\text{or } b - (x^2 + y^2)^{\frac{1}{2}} = \left\{a^2 - k^2 + \frac{2k}{p'}(c - px)\right\}^{\frac{1}{2}}.$$

COR. This is an equation to an epicycloid, as might be shewn. We shall, however, instead of this, shew geometrically that an epicycloid will satisfy the conditions. An epicycloid is the figure described by a point in one circle which *rolls* upon the circumference of another circle, which is fixed.

Let CP', fig. 102, be the radius of the given circle, and K the pully in the vertical line CK. In this line produced take a point O, so that $CK : KO ::$ weight P' : weight P; and in the same line take Oq equal to the length of the string $P'KP$. Take qs equal to qO, and qp equal to qK; and describe a circle qr with center s and radius sq. Let this circle, carrying along with it the point p in the radius sq, produced if necessary, roll along the circle described with center O and radius Oq: the point p will describe a curve pKP, which will possess the property required.

For let qs come into the position QS, so that the describing point p may come to P: if T be the point where the circles are in contact, the circle SQ may, for an instant, be supposed to revolve about the point T, so that the curve will be perpendicular to TP; hence the re-action of the curve will be in the direction PT.

Let SP produced meet CK in y, and let KP be joined: and since, by the description of the curve, the arc TQ is equal to the arc Tq, the angle TSQ is equal to the angle TOq, and therefore yO equal to yS. Also SQ equals sq or Oq, and QP equals qp or qK; hence SP equals OK, and therefore yP equals yK. Hence KP is parallel to OS; and hence if PV be parallel to KO, PV will equal KO; also OV will equal KP.

We made $CK : KO :: P' : P$; hence if KC represent the weight P', OK or VP will represent the weight P. Now the weight P' is kept at rest by three forces, gravity, re-action, and tension, in the directions KC, CP', $P'K$; hence, on this supposition $P'K$ represents the tension of KP'. And the weight P is kept at rest by three forces, gravity, re-action, and tension, in the directions VP, PT, TV; hence TV represents the tension of KP.

Now OT equals KP and KP', and OV equals KP; therefore TV equals KP; and hence the tensions of KP and of KP' are equal, and the bodies will balance each other.*

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* If instead of supposing a weight P' to rest on the circumference of a circle, we suppose CP' a heavy mass, (as a draw-bridge,) moveable about a hinge at C; and if it be required to find the curve on which P must rest so as always to balance it, the question will easily be seen to be the same. Under this form the Problem was solved by the Marquis de l'Hopital in the Leipsig Acts for February 1695. The curve, which was at first called *the Curve of Equilibration*, was shewn by John Bernoulli to be such an epicycloid as we have proved it to be. From the construction it appears, that if Oq, the length of the string, be to KC as P to P', the curve is the common epicycloid, in which the describing point is in the circumference of the rolling circle or *rota:* if the former ratio be less, as in fig. 102, the describing point is without the circumference of the *rota;* if greater, the describing point is within the circumference of the *rota* and the curve has a point of contrary flexure, as in fig. 101.

The Problems we have solved in the text suggest the following:

PROB. V. *Two weights connected by a string passing over a fixed pully rest on the same curve; to find the nature of the curve that they may in all positions balance.*

By Art. 17, we must have

$$px + p'x' = c, \quad r + r' = b;$$

also r and r' must be the same function of x and x'; that is, if f represent this function,

$$r = f(x), \quad r' = f(x').$$

$$\text{Let } x = t + \frac{c}{p+p'}, \quad x' = t' + \frac{c}{p+p'}, \quad r = u + \frac{b}{2}, \quad r' = u' + \frac{b}{2};$$

whence our equations become

$$pt + p't' = 0, \quad u + u' = 0; \quad \therefore t' = -\frac{pt}{p'}; \quad u' = -u.$$

Also u and u' will be the same function of t and t';

$$\therefore \phi(t) = -\phi(t') = -\phi\left(-\frac{pt}{p'}\right);$$

from which the form of the function ϕ must be determined, whence the form of f and the nature of the curve will be known.

It does not appear that there exists a solution to this equation, when p and p' are unequal. If $p' = p$, it becomes

$$\phi(t) = -\phi(-t);$$

that is, ϕ must be such a function that it only changes its sign by putting $-t$ for t. This condition will manifestly be satisfied by the functions,

$\phi(t) = mt,$

$\phi(t) =$ any rational function composed of odd powers of t,

$\phi(t) = m \,.\, \sin.\, nt$, &c.

19. PROB. IV. *Fig.* 103. *A body P, which hangs by a string CP, without weight, and consequently must be somewhere in the circumference whose center is C, is sustained at the point P by a repulsion acting from the lowest point A. The repulsive force is directly proportional to the intensity of the repulsive power in A and inversely proportional to the square of the distance. Knowing the repulsive power, to find the position; and conversely, from the position, to find the intensity of the repulsive power.*

An instrument of this kind is used to measure the intensity of the electrical repulsions which exist between two bodies A and P, in the same state of electricity; and it is then called the *Electrometer*.

Let $CA = CP = a$, $AP = r$; NP, perpendicular on CA, $= y$; $AN = x$. And let f represent the intensity of the repulsive power of A: then the force which it exerts at the distance r will be proportional to $\frac{f}{r^2}$; and if f be equal to the force at a distance $= 1$, $\frac{f}{r^2}$ will be *equal* to the force

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If we make $\phi(t) = mt$, we have $u = mt$;

$$\text{or } r - \frac{b}{2} = mx - \frac{mc}{2p};$$

$$\text{or } (x^2 + y^2)^{\frac{1}{2}} = mx - \frac{b}{2} - \frac{mc}{2p}$$

which will give a hyperbola as in Prob. V. In fact, it is manifest that since equal weights in two positions P and P', would each support a weight Q, they will support each other; and this in every situation.

If we make $\phi(t) = m . \sin. nt$, we have $u = m . \sin. nt$;

$$\text{or } r - \frac{b}{2} = m . \sin. n\left(x - \frac{c}{2p}\right):$$

and if we now suppose, that when $x' = 0$, x is $= 2h$, we have $c = 2ph$; and hence

$$r = \frac{b}{2} + m \sin\!: n(x - h).$$

When $x = h$, $r = \frac{b}{2}$; $\therefore r' = \frac{b}{2}$, and $x' = h$. Hence if with radius $KB = \frac{b}{2}$, fig. 104, we describe a circle, and take $KH = h$ in the vertical line, and draw DH horizontal, D, D, will be corresponding positions of the weights. When one is at E the other will be at F; and in other positions P, P' they will rest on such a curve as is represented in the figure.

at P, in the direction AP. Resolve this force in the directions AN, NP, or PX, PY, and the forces will be

$$\frac{f}{r^2}\cdot\frac{x}{r}, \text{ and } \frac{f}{r^2}\cdot\frac{y}{r}.$$

Hence, considering also the action of gravity $= p$, we have, (see Art. 13.)

$$X = \frac{fx}{r^3} - p, \quad Y = \frac{fy}{r^3}.$$

But, in the circle, $y^2 = 2ax - x^2$; $\therefore\ y\dfrac{dy}{dx} = a - x$.

Hence the formula $X + Y\dfrac{dy}{dx} = 0$, or $Xy + Yy\dfrac{dy}{dx} = 0$,

becomes $Xy + Y(a - x) = 0$.

And putting for X and Y their values;

$$\frac{fxy}{r^3} - py + \frac{fay}{r^3} - \frac{fyx}{r^3} = 0,$$

or $\dfrac{fa}{r^3} = p$; $\therefore\ r = \left(\dfrac{fa}{p}\right)^{\frac{1}{3}}$; whence the position is known.

And $\dfrac{f}{p} = \dfrac{r^3}{a}$, whence the ratio of the force f to the weight p is known.

Cor. If another force of the same kind, $(= f')$ balance the same body P, at a distance r' from A, we have also

$$\frac{f'}{p} = \frac{r'^3}{a}; \quad \therefore\ \frac{f'}{f} = \frac{r'^3}{r^3};$$

or the forces are as the cubes of the distances from A at which the body is supported.

The Principle of Virtual Velocities.

20. In the Elementary Treatise, Articles 45 to 57, the following proposition was demonstrated, upon the suppositions which were there adopted:

In any of the simple machines, the power is to the weight, as the weight's velocity in the direction of its action is to the power's velocity in the direction of its action.

This proposition was proved by shewing its truth in the case of each of the mechanical powers separately. In some of the cases, the power was supposed to act in a direction parallel to that in which the weight acts, both forces being supposed to be the result of gravity; as in the case of the toothed-wheels, Art. 48, and the pullies, Art. 49, &c. But this supposition does not affect the result; and if we suppose the attractive force which produces the power in these cases to act in a direction different from the direction of gravity which acts on the weight, the proposition will still appear to be true by the same reasoning. The string by which the power acts, will be drawn in the direction of the accelerating force from which the power results, and both the power itself and the space described by the power, corresponding to a given space described by the weight, will be the same as they were in the demonstration to which we refer; and therefore the validity of the demonstration will not be disturbed by such a change of supposition.

21. But any number of points connected in any manner may be considered as a machine, and the proposition thus proved for simple machines is capable of being extended to this more general case. Each of the points may be so constrained in its motions by strings, rods, and surfaces, that it is not at liberty to move otherwise than in a certain curve or curve surface: and any two of the points may be connected by strings or rods so as to affect each other's motions. The construction of the machine may be such, that the strings may have to move through fixed rings or over fixed pullies; and in this way the motion of one point will constrain the motion of one or more of the others; and there will be mutual forces exerted among the points. But these mutual forces must be such as destroy each other: for instance, if two points be connected by a string, the pressures which each point exerts upon the other, by means of the string, must be equal and opposite: and the same will be true of the forces exerted by means of rigid rods. Now there may be obtained a relation among the external forces which act on the machine, independent of the internal forces or mutual pressures; and the expression of this relation forms the ex-

tension of the proposition in Art. 45. of the Elementary Treatise of which we have spoken.

This proposition thus extended is called *the Principle of Virtual Velocities;* it may be expressed as follows.

22. Let P, Q, R, &c. be any forces which, acting at certain points of any machine whatever, balance each other; and let α, β, γ, &c. be respectively the spaces through which the points at which these forces act can move simultaneously, the relation of these spaces being determined by the construction of the machine, and the spaces being considered as negative, when the points move in directions opposite to the forces. This being supposed, and the spaces α, β, γ, &c. having the ratio to which they tend when they are indefinitely diminished, we shall have

$$P\alpha + Q\beta + R\gamma + \&c. = 0.$$

In order to prove this, let it be observed in the first place, that α, β, γ, &c. the spaces described by the points at which the forces act, are, when they are indefinitely diminished, as the velocities of these points. We shall suppose them to be thus diminished; and shall call them the *virtual velocities* of the points to which they belong.

If we had only *two* forces, P and Q, acting on a machine, we should have, by Art. 45, of the Elem. Tr.

$$P : Q :: -\beta : \alpha;$$

one of the quantities α, β, being made negative, because if the one point moves in the direction of the force which acts upon it, the other point must move in a direction opposite to the force which acts upon it. Hence

$$P\alpha = -Q\beta; \text{ and } P\alpha + Q\beta = 0.$$

Suppose now that three forces P, Q, R, balance each other, and that α, β, γ are their virtual velocities. Let $Q = Q' + Q''$, where Q' is of such a magnitude that $P\alpha + Q'\beta = 0$; therefore, by the former case, the force Q' will balance P; and since the three P, Q, R, are in equilibrium, the remaining force Q''

must balance R; whence we have also $Q''\beta + R\gamma = 0$; and adding this to the former equation, we have

$$Pa + (Q' + Q'')\,\beta + R\gamma = 0; \text{ or}$$
$$Pa + Q\beta + R\gamma = 0.$$

If there be four forces P, Q, R, S, which balance each other, and if a, β, γ, δ be the virtual velocities, as before; let $Q = Q' + Q''$; where $Pa + Q'\beta = 0$; and let $R = R' + R''$, where $Q''\beta + R'\gamma = 0$; then the part Q' of the force Q will balance P; the remaining part of Q, namely Q'', will balance R'; and hence the remaining part of R, namely R'', must balance S. Hence we must have, by the former case,

$$R''\gamma + S\delta = 0.$$

Adding to this the former equations

$$R'\gamma + Q''\beta = 0;$$
$$Pa + Q'\beta = 0;$$

we have $Pa + (Q' + Q'')\,\beta + (R' + R'')\,\gamma + S\delta = 0$,

or $Pa + Q\beta + R\gamma + S\delta = 0$.

And in the same manner we may form a similar equation whatever be the number of forces*.

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* This is the proof of the Principle of Virtual Velocities which is given by Lagrange in the *Mecanique Analytique.* To make the proof complete, the point at which the force Q' acts should be subject to the same constraint in the system P, Q', as the point at which Q acts, in the system P, Q, R: and similarly for Q''. If the point at which Q acts be constrained to move on a given line, this condition is fulfilled. But if this point be free to move in any direction, we may, in assuming the systems P, Q', and Q'', R, suppose the point constrained to move in the direction of the force Q. For the point on which Q' acts may be acted on by a force M, perpendicular to the line in which Q' acts, so that it shall be constrained to move in that line. If the point, when kept at rest by M, were to move through a small space μ, perpendicular to the line of Q's direction, we should have

$$Pa + M\mu = 0,$$

by the former case. In the same manner if M' be the force which would prevent the point at which Q'' acts from leaving the line of the direction of Q''; and if, when the point is acted on by M', it moves through a small space μ, in the direction of this force, we shall have

$$M'\mu + R\gamma = 0.$$

Hence

$$Pa + (M + M')\,\mu + R\gamma = 0.$$

This formula expresses *the Principle of Virtual Velocities.* It enables us to put into an equation the conditions of equilibrium of any system of forces whatever, anyhow connected.

For this purpose we may consider two possible positions of the system indefinitely near each other; and find general expressions for α, β, γ, &c. introducing into these expressions as many indeterminate quantities as there are arbitrary elements in the variation of position of the system. These expressions are to be substituted for α, β, γ, δ in the above equation. And this equation must be true independently of all the indeterminate quantities, in order that the equilibrium may subsist in general, and that motion may not take place in any direction. We must therefore take the sum of the terms which involve each of the indeterminate quantities, and make each sum separately equal to nothing. In this way we shall have as many distinct equations as there are indeterminate quantities. This will also be the same number as that of the unknown quantities in the position of the system; we shall therefore have thus as many equations as are requisite to determine the position of the system.

But when the point at which Q acts moves through a space perpendicular to the direction of Q, we may suppose the motion of this point to be constrained, and $\beta = 0$; whence

$$P\alpha + R\gamma = 0,$$

by what has been already said. Therefore

$$M + M' = 0,$$

or the forces which constrain the points at which Q', Q'' act, in the systems P, Q', and Q'', R, destroy each other when the two systems are combined; and therefore the point at which $Q' + Q''$ acts, is then acted upon by that force, that is, by the force Q, and by no other; whence the demonstration above given is valid.

CHAP. III.

THE CONDITIONS OF EQUILIBRIUM OF A RIGID BODY.

23. PROP. *To find the resultant of any number of parallel forces acting on a rigid body. Fig.* 105.

Let any number of parallel forces p_1, p_2, p_3, &c. act upon a rigid body. Let a plane yAx be drawn, perpendicular to these forces; and let two lines, Ax, and Ay, be drawn in this plane at right angles to each other. Let P_1M_1 be parallel to Ay, and let x_1, y_1 be AM_1, M_1P_1, the *co-ordinates* of the point P_1, where p_1 meets the plane yAx. Similarly let x_2, y_2, be the co-ordinates of P_2, where p_2 meets the plane; x_3, y_3, the same quantities for P_3, &c. And let R be the resultant of the forces, and α, β, the co-ordinates of the point where it meets the plane.

The two forces p_1, p_2, produce the same effect as if they acted at P_1, P_2. And if we consider them as weights, they will balance each other upon their center of gravity, and produce at that point a pressure $= p_1 + p_2$. (*Elem. Tr.*) Hence their effect upon the rigid body is the same as that of a force $p_1 + p_2$ acting at the center of gravity of P_1, P_2, in a direction parallel to these forces. Similarly it will appear that this force $p_1 + p_2$, along with p_3, that is, p_1, p_2, p_3, will produce the same effect as $p_1 + p_2 + p_3$, acting at the center of gravity of P_1, P_2, P_3. And in the same manner it may be shewn that any number of forces p_1, p_2, p_3, &c. will produce the same effect as $p_1 + p_2 + p_3 +$ &c. acting parallel to the forces, at the center of gravity of P_1, P_2, P_3, &c.

Hence we shall have, by the properties of the center of gravity (*Elem. Tr.*)

$$R \;= p_1 + p_2 + p_3 + \text{\&c.}$$
$$R\alpha = p_1x_1 + p_2x_2 + p_3x_3 + \text{\&c.}$$
$$R\beta = p_1y_1 + p_2y_2 + p_3y_3 + \text{\&c.}$$

Whence R is known, and a, β, which determine the position of the resultant.

Cor. 1. If any of the forces act in the opposite direction they must be considered as negative.

Hence it appears that we may have $p_1 + p_2 + p_3 +$ &c. $= 0$. In this case, if $p_1 x_1 + p_2 x_2 +$ &c. be finite, a will be infinite. And similarly for β.

Cor. 2. For example, let two forces, each $= p$, act in opposite directions at points in the line Ax, distant from each other by a distance a. Hence we shall have

$$R = 0; \quad Ra = p(x_1 + a) - px_1 = pa;$$

$$\beta = 0; \quad a = \frac{pa}{R} = \frac{pa}{0}.$$

Therefore a is infinite, and the forces are equivalent to a force $= 0$, acting at any infinite distance.

In this case no single force could produce the effect of the two. Their tendency is to turn the system round in the plane in which they are, without producing any motion except a rotatory one.

Cor. 3. Hence it is not true that parallel forces can in all cases be reduced to a single finite force. If $p_1+p_2+p_3+$&c.$=0$, they can not.

In this case the forces can be reduced to two, equal to each other, and acting at two different points in opposite directions. For since $p_1 + p_2 +$ &c. $= 0$, these forces may be divided into two groups, of which one is equal to the other with a negative sign. And hence if we take the resultants of these groups separately, we shall obtain two equal forces in opposite directions.

Cor. 4. Let one of these groups consist of p_1, p_2, &c. and the other of p', p'', &c. Then

$$p_1 + p_2 + \text{\&c.} + p' + p'' + \text{\&c.} = 0,$$
$$p_1 + p_2 + \text{\&c.} = -p' - p'' - \text{\&c.}$$

And if R be the resultant of p_1, p_2, &c., $-R$ will also be the resultant of p', p'', &c. Let α, β, be the co-ordinates of the point where the first resultant meets the plane, α', β', the corresponding co-ordinates for the second resultant. Then

$$R\alpha = p_1x_1 + p_2x_2 + \text{\&c.}; \quad R\beta = p_1y_1 + p_2y_2 + \text{\&c.}$$
$$-R\alpha' = p'x' + p''x'' + \text{\&c.}; -R\beta' = p'y' + p''y'' + \text{\&c.}$$
$$\therefore R(\alpha - \alpha') = p_1x_1 + p_2x_2 + \text{\&c.} + p'x' + p''x'' + \text{\&c.}$$
$$R(\beta - \beta') = p_1y_1 + p_2y_2 + \text{\&c.} + p'y' + p''y'' + \text{\&c.}$$

And the quantities on the right hand are the same whatever be R, that is, however the groups are selected. If l be the line which joins the two points of application, λ the angle which it makes with Ax,

$$Rl = R\sqrt{\{(\alpha - \alpha')^2 + (\beta - \beta')^2\}}; \ \tan.\lambda = \frac{\beta - \beta'}{\alpha - \alpha'};$$

and these quantities are the same whatever R be.

Hence the position of the line l, and the moment of the pair of forces to turn the system in the plane in which are R and l, are found to be the same, however the two groups are selected.

24. Prop. *To find the conditions of equilibrium of parallel forces acting upon a rigid body.*

In order that the equilibrium may subsist, one of the forces, as p_1, must be equal and opposite to the resultant of all the others. Hence $-p_1$ must be the resultant of the forces p_2, p_3, &c. And therefore by last Article, x_1, y_1, x_2, y_2, &c. being the co-ordinates, as before,

$$-p_1 = p_2 + p_3 + \text{\&c.}$$
$$-p_1x_1 = p_2x_2 + p_3x_3 + \text{\&c.}$$
$$-p_1y_1 = p_2y_2 + p_3y_3 + \text{\&c.};$$
$$\therefore p_1 + p_2 + p_3 + \text{\&c.} = 0 \ldots\ldots\ldots\ldots\ldots (1);$$
$$\left.\begin{array}{l} p_1x_1 + p_2x_2 + p_3x_3 + \text{\&c.} = 0 \\ p_1y_1 + p_2y_2 + p_3y_3 + \text{\&c.} = 0 \end{array}\right\} \ldots\ldots\ldots (2).$$

COR. 1. If the rigid body have one point fixed, let this point be the origin of co-ordinates. And it is manifest that the equilibrium will subsist if the resultant pass through this point; for it will be counteracted by the resistance of the fixed point. Hence in last Article $a = 0$, $\beta = 0$. Therefore, by that Article,

$$p_1x_1 + p_2x_2 + p_3x_3 + \&c. = 0;$$

$$p_1y_1 + p_2y_2 + p_3y_3 + \&c. = 0.$$

25. PROP. *To find the resultant of any number of forces acting in the same plane upon a rigid body. Fig.* 106.

Let p_1, p_2, p_3, &c. be the forces acting in the plane yAx, at the points P_1, P_2, P_3, &c.: let Ax, Ay be at right angles in this plane; and let x_1, y_1; x_2, y_2; x_3, y_3, &c. be the co-ordinates of these points parallel to Ax, Ay. Let P_1D_1, P_1E_1 be lines parallel to Ax, Ay, and let a_1, be the angle which P_1p_1 makes with P_1D_1. If the force p_1 be resolved in these directions, the components will be p_1 cos. a_1, p_1 sin a_1. In the same manner p_2 cos. a_2, p_2 sin. a_2 will be the components of p_2; and similarly for the others. Hence the forces are thus resolved into two sets of parallel forces, acting at the points P_1, P_2, P_3, &c., and parallel to Ax and to Ay respectively. Let X be the resultant of the first set; Y, of the second. Also let X meet Ay in K, and let $AK = t$; and let Y meet Ax in H, and let $AH = s$. Then we may suppose X to act at K, and Y at H. Therefore, by Art. 23,

$$X = p_1 \cos. a_1 + p_2 \cos. a_2 + p_3 \cos. a_3 + \&c.$$

$$Y = p_1 \sin. a_1 + p_2 \sin. a_2 + p_3 \sin. a_3 + \&c.$$

$$Ys = p_1x_1 \sin. a_1 + p_2x_2 \sin. a_2 + p_3x_3 \sin. a_3 + \&c.$$

$$Xt = p_1y_1 \cos. a_1 + p_2y_2 \cos. a_2 + p_3y_3 \cos. a_3 + \&c.$$

Hence we know X, Y, s, t. And knowing AK, AH, if we draw KG, HG parallel to Ax, Ay, the forces X, Y may be supposed to act at G; and will then produce a resultant R, which is the resultant of the whole system. Also if a be the angle which this resultant makes with Ax, we shall have

$$R = \sqrt{(X^2 + Y^2)}; \ \tan. a = \frac{Y}{X}.$$

Hence we know the magnitude and position of R. The co-ordinates s, t, of its point of application have already been found.

Cor. 1. To find the equation of the straight line in which R acts.

Let the equation to the straight line be $y = Ax + B$. In this case A is the tangent of the angle a; therefore $A = \frac{Y}{X}$,

$$\text{and } y = \frac{Y}{X}x + B.$$

Also, since G is a point in this line, when $x = s$, $y = t$.

Therefore $t = \frac{Y}{X}s + B$; and $B = t - \frac{Y}{X}s$. Hence

$$y = \frac{Y}{X}x + t - \frac{Y}{X}s; \text{ and } Xy - Yx = Xt - Ys,$$

which is the equation to the line.

Cor. 2. Putting for Xt and Ys their values, we have

$$Xy - Yx = p_1 (y_1 \cos. a_1 - x_1 \sin. a_1)$$
$$+ p_2 (y_2 \cos. a_2 - x_2 \sin. a_2) + \&c.$$

call this quantity L. Then the equation is

$$Xy - Yx = L.$$

Cor. 3. In this case, when *one* of the sets of parallel forces, as that parallel to Ax, is not reducible to a single force, (see Cor. 3, Art. 23,) we shall have $X = 0$, $t = \text{inf.}$ Hence

$$R = Y, \ \cos. a = 0, \ s = \frac{p_1 x_1 \sin. a_1 + p_2 x_2 \sin. a_2 + \&c.}{Y}.$$

Hence the resultant will be a single force parallel to Ay, and determined in magnitude and position by the above equations.

Cor. 4. But in the case when *both* the sets of parallel forces are incapable of being reduced to single forces, we shall have $X = 0$, $Y = 0$, $s = \text{inf.}$ $t = \text{inf.}$ Hence $R = 0$; and we have a resultant $= 0$, acting at an infinite distance.

In this case the forces are equal to two, equal and opposite, but not in the same line, as in Cor. 3, Art. 23.

26. Prop. *To find the conditions of equilibrium of any number of forces acting in the same plane upon a rigid body.*

In order that there may be an equilibrium, p_1 must be equal and opposite to the resultant of p_2, p_3, &c. Hence $-p_1$ must be the resultant of p_2, p_3, &c. And $-p_1 \cos. a_1$, $-p_2 \sin. a_2$ will be the parts of the resultant which are parallel to Ax, Ay. Hence, by the equations of Art. 23,

$$-p_1 \cos. a_1 = p_2 \cos. a_2 + p_3 \cos. a_3 + \&c.$$

$$-p_1 \sin. a_1 = p_2 \sin. a_2 + p_3 \sin. a_3 + \&c.$$

Hence

$$\left.\begin{array}{l} p_1 \cos. a_1 + p_2 \cos. a_2 + p_3 \cos. a_3 + \&c. = 0 \\ p_1 \sin. a_1 + p_2 \sin. a_2 + p_3 \sin. a_3 + \&c. = 0 \end{array}\right\} \ldots\ldots (1).$$

Also the equation of the line in which $-p_1$ acts must be the same as that in which the resultant of p_2, p_3, &c. acts. And the latter equation is, (Cor. 2, Art. 25,) putting for X, $-p_1 \cos. a_1$, and for Y, $-p_1 \sin. a_1$,

$$-p_1 y \cos. a_1 + p_1 x \sin. a_1 = p_2 (y_2 \cos. a_2 - x_2 \sin. a_2)$$
$$+ p_3 (y_3 \cos. a_3 - x_3 \sin. a_3) + \&c.$$

And this, which is true for every point of p_1's direction, must be true for the point P_1, where $x = x_1$, $y = y_1$. Hence, putting these values for x and y, and transposing,

$$p_1 (y_1 \cos. a_1 - x_1 \sin. a_1) + p_2 (y_2 \cos. a_2 - x_2 \sin. a_2)$$
$$+ p_3 (y_1 \cos. a_3 - x_3 \sin. a_3) + \&c. = 0 \ldots\ldots\ldots (2).$$

This equation (2) and the two found above (1) are the equations of condition for the equilibrium of the forces.

Cor. 1. If a point of the rigid body, in the plane in which the forces act, be a fixed point, the equilibrium will subsist if the resultant of the forces pass through this point; for the effect of the force will be balanced by the re-action of the fixed point.

Let A be the fixed point. Then, in order that the resultant may pass through A, we must have $x = 0$, $y = 0$, at the same time;

$$\therefore 0 = Xt - Ys,$$

$$\text{or } 0 = p_1 (y_1 \cos. a_1 - x_1 \sin. a_1) + p_2 (y_2 \cos. a_2 - x_2 \sin. a_2) + \&c.$$

which is in this case the condition of equilibrium.

27. Prop. *Any number of forces being given, acting in any directions upon a rigid body, to reduce them to two sets of forces, one set being in a given plane, and the other perpendicular to it.*

Let Ax, Ay, Az, fig. 107, be three lines at right angles to each other. Let P be a point of the system, at which one of the forces acts, in the direction Pp. Let PD, PE, PF be three lines parallel to Ax, Ay, Az, and let a, β, γ be the angles which Pp makes with PD, PE, PF. Then p being the force, $p \cos. a$, $p \cos. \beta$, $p \cos. \gamma$ will be the components in PD, PE, PF. Let FP meet the plane yAx in O, and let OM and ON be parallel to Ay and Ax.

If we suppose, at the point P, two equal forces in opposite directions to be added to the system, these will counteract each other, and the effect of the forces such as p will be the same as before. Let two forces, g in the direction PF, and g in the direction FP, act at P. Then the forces which act at P may be grouped thus,

$$p \cos. a \text{ and } g; \quad p \cos. \beta \text{ and } -g; \quad p \cos. \gamma.$$

Let p cos. a in PD, and g in PF have a resultant Ph: hP will be in the plane DPF: let it meet ON in H. Ph acting at P is equivalent to Ph acting at H. And Ph at H may be resolved in two forces, p cos. a parallel to Ax, and g parallel to Az.

Since Ph is compounded of p cos. a in the direction HO, and g in the direction OP, we shall have, calling the co-ordinates of P, x, y and z,

$$HO : OP (= z) :: p \cos. a : g;$$

$$\therefore HO = \frac{pz \cos. a}{g}. \quad \text{And } NH = NO - HO = x - \frac{pz \cos. a}{g}.$$

Hence p cos. a and g at P, are equivalent to g parallel to Az, and p cos. a parallel to Ax; both acting at a point H, of which the co-ordinates parallel to

$$Ax \text{ and } Ay, \text{ are } x - \frac{pz \cos. a}{g}, \text{ and } y.$$

In the same manner p cos. β and $-g$ are equivalent to a force Pk in the plane EPF, and this produces the same effect as if it acted at K. And at K it may be resolved into

$$p \cos. \beta \text{ and } -g.$$

Also as before,

$$OK = \frac{pz \cos. \beta}{g}; \text{ and } MK = y + \frac{pz \cos. \beta}{g}.$$

Hence p cos. β and $-g$ are equivalent to p cos. β parallel to Ay and $-g$ parallel to Az; both acting at a point K of which the co-ordinates parallel to

$$Ax \text{ and } Ay, \text{ are } x \text{ and } y + \frac{pz \cos. \beta}{g}.$$

p cos. γ is parallel to Az, and produces the same effect as if it acted at O, of which the co-ordinates are x, y.

Hence if p_1 is a force acting at a point of which the co-ordinates are x_1, y_1, z_1, making with the three co-ordinate angles a_1, β_1, γ_1; and if p_2, p_3, &c. be other forces; x_2, y_2, z_2; x_3, y_3, z_3, &c. the corresponding co-ordinates; a_2, β_2, γ_2; a_3, β_3, γ_3, &c. the corresponding angles; the forces p_1, p_2, p_3, &c. will be equivalent to the following forces in the plane yAx;

$p_1 \cos. \alpha_1$, par. to Ax, with co-ordinates $x_1 - \dfrac{p_1 z_1 \cos. \alpha_1}{g_1}$ and y_1;

$p_1 \cos. \beta_1$, par. to Ay, with co-ordinates x_1 and $y_1 + \dfrac{p_1 z_1 \cos. \beta_1}{g_1}$;

and to the following forces parallel to Az,

$p_1 \cos. \gamma_1$ with co-ordinates x_2, y_1;

g_1 with co-ordinates $x_1 - \dfrac{p_1 z_1 \cos. \alpha_1}{g_1}$ and y_1;

$- g_1$ with co-ordinates x_1 and $y_1 + \dfrac{p_1 z_1 \cos. \beta_1}{g_1}$.

And to similar forces with the exponents 2, 3, &c. instead of 1.

28. PROP. *To find the conditions of equilibrium of any number of forces p_1, p_2, p_3, &c. acting in any directions upon a rigid body.*

The forces being resolved as in the last Article, the equilibrium will subsist if the forces in the plane yAx, and the forces parallel to Az be in equilibrium separately. Hence we shall have by Art. 26, these three equations for the equilibrium of the forces in the plane yAx:

$$\left.\begin{array}{l} p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 + \&c. = 0 \\ p_1 \cos. \beta_1 + p_2 \cos. \beta_2 + p_3 \cos. \beta_3 + \&c. = 0 \end{array}\right\} \ldots\ldots (1);$$

$$p_1 (y_1 \cos. \alpha_1 - x_1 \cos. \beta_1) + p_2 (y_2 \cos. \alpha_2 - x_2 \cos. \beta_1)$$
$$+ p_3 (y_3 \cos. \alpha_3 - x_3 \cos. \beta_3) + \&c. = 0 \ldots\ldots\ldots (2).$$

Also by Art. 24, we have, for the equilibrium of the forces parallel to Az,

$$p_1 \cos. \gamma_1 + \&c. = 0;$$

$$p_1 x_1 \cos. \gamma_1 + g_1 \left(x_1 - \frac{p_1 z_1 \cos. \alpha_1}{g_1}\right) - g_1 x_1 + \&c. = 0;$$

$$p_1 y_1 \cos. \gamma_1 + g_1 y_1 - g_1 \left(y_1 - \frac{p_1 z_1 \cos. \beta_1}{g_1}\right) + \&c. = 0;$$

with other similar terms corresponding to p_2, g_2, p_3, g_3, &c. And these three latter equations become

$$p_1 \cos. \gamma_1 + p_2 \cos. \gamma_2 + p_3 \cos. \gamma_3 + \&c. = 0 \ldots\ldots\ldots (1);$$

$$\left.\begin{aligned} p_1 (x_1 \cos. \gamma_1 - z_1 \cos. \alpha_1) + p_2 (x_2 \cos. \gamma_2 - z_2 \cos. \alpha_2) \\ + p_3 (x_3 \cos. \gamma_3 - z_3 \cos. \alpha_3) + \&c. = 0, \\ p_1 (y_1 \cos. \gamma_1 - z_1 \cos. \beta_1) + p_2 (y_2 \cos. \gamma_2 - z_2 \cos. \beta_2) \\ + p_3 (y_3 \cos. \gamma_3 - z_3 \cos. \beta_3) + \&c. = 0. \end{aligned}\right\} \ldots (2).$$

which these three equations, with the former three, are the conditions of equilibrium.

Cor. 1. It has been proved that these equations are *sufficient;* that is, that if they are satisfied the equilibrium subsists. They are also *necessary;* for except both sets are satisfied the equilibrium does not subsist.

If possible, let the equilibrium subsist when the forces parallel to z are not separately in equilibrium. The equilibrium will still subsist if we suppose any line in the plane yAx to be fixed. But in that case, all the forces in the plane yAx will be counteracted by the resistance of this line. And the forces parallel to Az will turn the system about this line in some of its positions. Hence the equilibrium will not subsist.

And since the forces parallel to Az are in equilibrium separately, the other forces must also be in equilibrium separately.

Cor. 2. Let the rigid body be moveable about a fixed point. Let this point be made the origin of co-ordinates A. Then the forces may be resolved as in Art. 27. and the equilibrium will subsist if the forces in the plane yAx have a resultant which passes through the point A, and if the forces parallel to Az have also a resultant which passes through A. Hence by Art. 25, Cor. 1, and by Art. 24, Cor. 1, we have

$$p_1 (y_1 \cos. \alpha_1 - x_1 \cos. \beta_1) + p_2 (y_2 \cos. \alpha_2 - x_2 \cos. \beta_2) + \&c. = 0;$$

$$p_1 (x_1 \cos. \gamma_1 - z_1 \cos. \alpha_1) + p_2 (x_2 \cos. \gamma_2 - z_2 \cos. \alpha_2) + \&c. = 0;$$

$$p_1 (y_1 \cos. \gamma_1 - z_1 \cos. \beta_1) + p_2 (y_2 \cos. \gamma_2 - z_2 \cos. \beta_2) + \&c. = 0.$$

It appears from this, that the forces are to be such as to keep each other in equilibrium about three axes at right angles to each other passing through the fixed point.

29. Prop. *To find the condition which is requisite in order that a system of forces acting anyhow in space may have a single resultant.*

Retaining the notation of Art. 27, we may reduce the forces to the two sets mentioned in that Article. The resultants of these sets may be found by Articles 23 and 25; and if these resultants intersect each other, they may be compounded into a single force which will be the resultant of the whole. If the two resultants do not intersect each other, this will be impossible.

$$\text{Let } p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + \&c. = X;$$
$$p_1 \cos. \beta_1 + p_2 \cos. \beta_2 + \&c. = Y;$$
$$p_1 \cos. \gamma_1 + p_2 \cos. \gamma_2 + \&c. = Z.$$
$$p_1 (y_1 \cos. \alpha_1 - x_1 \cos. \beta_1) + p_2 (y_2 \cos. \alpha_2 - x_2 \cos. \beta_2) + \&c. = L;$$
$$p_1 (x_1 \cos. \gamma_1 - z_1 \cos. \alpha_1) + p_2 (x_2 \cos. \gamma_2 - z_2 \cos. \alpha_2) + \&c. = M;$$
$$p_1 (z_1 \cos. \beta_1 - y_1 \cos. \gamma_1) + p_2 (z_2 \cos. \beta_2 - y_2 \cos. \gamma_2 + \&c. = N^*.$$

Then we shall have for the equation of the line in which the force acts, which is the resultant of those in the plane yAx,

$$Xy - Yx = L; \text{ (Cor. 2, Art. 25.)}$$

and for the force which is the resultant of those parallel to Az we shall have, by Art. 23, (as in Art. 28.)

$$Zx = M; \quad Zy = -N.$$

In order that these two forces may intersect, the point in which the latter meets the plane yAx must be in the line of the direction of the former. Hence the equation $Xy - Yx = L$

* The quantities L, M, N are the *moments* of the forces p_1, p_2, &c. *projected* on the planes xy, xz, yz respectively. These projected moments give rise to some remarkable propositions. See Poisson, *Traité de Mec.* Liv. III. Chap. II.

must be satisfied by the values of x and y in $Zx = M$, $Zy = -N$; and substituting, we find

$$LZ + MY + NX = 0 \ldots\ldots (a),$$

which is the equation of condition required.

30. Prop. *In the case where it is possible, to find the resultant of any number of forces acting anyhow in space.*

The force in the plane yAx will be composed of X and Y, and the force at the same point parallel to Az will be Z. Hence, if R be the resultant, and a, b, c, the angles which it makes with lines parallel to Ax, Ay, Az, we shall have, as in Art. 11,

$$R = \sqrt{(X^2 + Y^2 + Z^2)};$$

$$\cos. a = \frac{X}{R}; \ \cos. b = \frac{Y}{R}; \ \cos. c = \frac{Z}{R}.$$

And the point where the resultant cuts the plane yAx is known by the equations $Zx = M$, $Zy = -N$.

Cor. It may be easily shewn that the equations to the line in which the resultant acts are

$$Xy - Yx = L, \ Zx - Xz = M, \ Yz - Zy = N \ldots\ldots (b),$$

of which two only are necessary, the third being included in them in consequence of the equation of condition (a) of last Article.

31. Prop. *In the case where a number of forces are not reducible to one force, they are always reducible to two.*

Without altering the conditions of the system, we may suppose, in addition to the forces of the system, two new forces S, $-S$, acting at the origin A, and making angles a, b, c, with the axes. And these forces and their angles may be so taken that the force S, along with p_1, p_2, &c., shall satisfy the equation (a), and have a single resultant. Thus the forces are reduced to this resultant, and to the force $-S$ acting at the point A.

Cor. In this case the two forces to which the system is reduced are not determined in magnitude and direction.

32. The following example may serve to illustrate the preceding Articles.

ABCDEFG, fig. 108, is a rectangular parallelepiped acted on by forces, which have their directions in the edges *BE*, *CF*, *DG* of the parallelepiped, taken so that none of them pass through *A*, and no two of them are in the same plane: to shew when there is a single resultant, and to find it.

Let AD, AB, AC be in the directions of Ax, Ay, Az; let $AD = a$, $AB = b$, $AC = c$: and let the forces be p_1, p_2, p_3. Then we shall have

$$p_1 \cos. \alpha_1 = p_1, \quad p_1 \cos. \beta_1 = 0, \quad p_1 \cos. \gamma_1 = 0;$$

$$p_2 \cos. \alpha_2 = 0, \quad p_2 \cos. \beta_2 = p_2, \quad p_2 \cos. \gamma_2 = 0;$$

$$p_3 \cos. \alpha_3 = 0, \quad p_3 \cos. \beta_3 = 0, \quad p_3 \cos. \gamma_3 = p_3.$$

$$x_1 = 0, \quad y_1 = b, \quad z_1 = 0;$$

$$x_2 = 0, \quad y_2 = 0, \quad z_2 = c;$$

$$x_3 = a, \quad y_3 = 0, \quad z_3 = 0;$$

Hence we have $X = p_1, \quad Y = p_2, \quad Z = p_3$

$$L = p_1 b; M = p_3 a, N = p_2 c.$$

And the equation of condition (a) becomes

$$p_1 p_3 b + p_2 p_3 a + p_1 p_2 c = 0,$$

$$\text{or } \frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3} = 0.$$

If this equation be not satisfied, the forces are not reducible to a single force.

Let p_1, p_2, be as the edges *BE*, *CF*. Then

$$\frac{b}{p_2} = \frac{a}{p_1}; \therefore \frac{c}{p_3} = -\frac{2a}{p_1} \text{ when the reduction is possible;}$$

$$\therefore L = p_1 b; \; M = p_3 a = -\tfrac{1}{2} p_1 c; \; N = p_2 c = p_1 \frac{bc}{a}.$$

Hence, by Cor. to Art. 30. the equations to the line of direction of the force will be

$$\left.\begin{aligned} p_1 y - p_2 x = p_1 b, &\text{ or } y - \frac{b}{a}x = b \\ p_3 x - p_1 z = -\tfrac{1}{2} p_1 c, &\text{ or } \frac{cx}{a} + 2z = c \end{aligned}\right\}.$$

These two equations determine the position of the force; and its magnitude is known, being

$$= \sqrt{(p_1^2 + p_2^2 + p_3^2)} = p_1 \frac{\sqrt{(a^2 + b^2 + \frac{1}{4}c^2)}}{a}.$$

If we produce DE to H, making $EH = ED$, BH will be the line to which the first equation belongs. And when

$$x = 0, \quad z = \tfrac{1}{2}c.$$

Hence, if we bisect BF in K, KH is the direction in which the resultant of the three forces acts.

CHAP. IV.

THE APPLICATION OF THE INTEGRAL CALCULUS TO FINDING THE CENTER OF GRAVITY.

33. Prop. *To find the center of gravity of any curvilinear body.*

Let $P'PBQQ'$ (fig. 109.) be any body: Ax the axis of x: and let the body be cut by planes PQ, $P'Q'$, perpendicular to Ax.

Let G, G', K, be the centers of gravity of the portions of the body PBQ, $P'BQ'$, and $PQQ'P'$; and let GH, $G'H'$, KL, be perpendiculars upon a plane Ay parallel to PM.

Let the mass $PBQ = m$, $P'BQ' = m'$; therefore we have $PQQ'P' = m' - m$.

Also let $GH = h$, $G'H' = h'$, $KL = k$.

Now we may suppose the masses PBQ, $PQQ'P'$, to be collected at their respective centers of gravity G, K: and since G' is the center of gravity of the whole mass, we have (*Elem. Tr.*)

$$G'H' = \frac{PBQ \,.\, GH + PQQ'P' \,.\, KL}{PBQ + PQQ'P'};$$

$$\text{or } h' = \frac{mh + (m' - m)\,k}{m'}.$$

$$\text{Hence, } k = \frac{m'h' - mh}{m' - m}.$$

Now, if we suppose the plane $P'Q'$ to come indefinitely near to PQ, so that $PQQ'P'$ may become an indefinitely thin slice, K will ultimately be in PQ, and k ultimately $= AM$ or x.

Also in this case $\dfrac{m'h' - mh}{m' - m}$, which is the ratio of the *increments* of mh and of m, will, by the principles of the differential calculus, ultimately become the ratio of their *differentials*. Hence taking the ultimate limits on both sides, which will necessarily be equal, we have

$$x = \frac{d \,.\, mh}{dm}.$$

$$\text{Hence, } \frac{d \,.\, mh}{dx} = \frac{d \,.\, mh}{dm}\,\frac{dm}{dx} = x\,\frac{dm}{dx};$$

$$\text{and integrating in } x,\ mh = \int_x x\,\frac{dm}{dx};$$

$$\therefore\ h = \frac{\int_x x\,\dfrac{dm}{dx}}{m}.$$

We may thus find the distance of the center of gravity from the known plane Ay.

If Ax, perpendicular to Ay, be a line along which abscissas are measured; and if $AM = x$, $MP = y$, the curve may be defined by a relation between x and y, if the body be a plane figure, or a figure of revolution round Ax, and hence $\frac{dm}{dx}$ may be found.

In other cases we may suppose two planes, at right angles to each other, passing through Ax; and if y and z be the distances of P from these planes, the surface of the body may be defined by an equation between x, y, and z, whence dm may be found.

If the body be symmetrical on the two sides of Ax, supposing it to lie in a plane; or if its section by every plane passing through Ax be symmetrical to Ax, supposing it extended in three dimensions; its center of gravity will be in the line Ax: and hence, to determine its position, it is sufficient to find the value of GH.

If the body be not thus symmetrical with respect to Ax, its center of gravity will not necessarily be in that line. In this case it will be necessary, if the body lie in a plane, to find the distance of the center of gravity from some other line besides Ay; for instance, to find GF its distance from Ax: this may be found in the same way as GH. If the body have three dimensions, it will be necessary to find, by similar methods, the distances of the center of gravity from three known planes, which will determine its position.

We shall consider the cases separately.

1. *A Symmetrical Area.*

34. Let PAp, fig. 110, be a curvilinear area symmetrical with regard to AM. We may suppose it to be a lamina of matter whose thickness may be neglected; or, if the thickness be supposed finite and constant, the position of G will be the same. It is manifest that the weight or quantity of matter of any part, supposing the density uniform, will be as the magnitude, or as the area of that part, and may be represented by the area.

Now if the abscissa AM (fig. 110,) $= x$, and the ordinate $MP = y$, $\frac{dm}{dx}$ is the differential of the area PAp, or $2y$; hence, if $GA = h$, we have, by Art. 33,

$$h = \frac{\int_x 2yx}{\int_x 2y} = \frac{\int_x xy}{\int_x y}.$$

We shall give some instances of the application of this formula.

Ex. 1. The equation to a curve is $y = \frac{x(a^2 - x^2)^{\frac{1}{2}}}{a}$; to find the center of gravity of its area,

$$\int_x y = \int_x \frac{x(a^2 - x^2)^{\frac{1}{2}}}{a} = C - \frac{(a^2 - x^2)^{\frac{3}{2}}}{3a};$$

C being an arbitrary constant; and if we suppose the area to be taken from the point where x and $y = 0$,

$$\int_x y = \frac{a^3 - (a^2 - x^2)^{\frac{3}{2}}}{3a}.$$

$$\int_x xy = \int_x \frac{x^2(a^2 - x^2)^{\frac{1}{2}}}{a} = \frac{\int_x x \,.\, x \,.\, (a^2 - x^2)^{\frac{1}{2}}}{a}.$$

But $\int_x x \,.\, x(a^2 - x^2)^{\frac{1}{2}} = -\frac{x \,.\, (a^2 - x^2)^{\frac{3}{2}}}{3} + \int_x \frac{(a^2 - x^2)^{\frac{3}{2}}}{3}$ *

$$= -\frac{x \,.\, (a^2 - x^2)^{\frac{3}{2}}}{3} + \int_x \frac{a^2 \,.\, (a^2 - x^2)^{\frac{1}{2}}}{3} - \int_x \frac{x^2 \,.\, (a^2 - x^2)^{\frac{1}{2}}}{3};$$

$$\therefore\ 4\int_x x^2 (a^2 - x^2)^{\frac{1}{2}} = -x \,.\, (a^2 - x^2)^{\frac{3}{2}} + a^2 \,.\, \int_x (a^2 - x^2)^{\frac{1}{2}}$$

$$= -x(a^2 - x^2)^{\frac{3}{2}} + \frac{a^2}{2} \,.\, \left\{x \,.\, (a^2 - x^2)^{\frac{1}{2}} + a^2 \,.\, \text{arc}\left(\sin. = \frac{x}{a}\right)\right\} + C;$$

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* By the formula $\int_x u\frac{dv}{dx} = uv - \int_x v\frac{du}{dx}$; see Lacroix's Elementary Treatise, &c. No. 148.

$$\therefore \int_x \frac{x^2 (a^2 - x^2)^{\frac{1}{2}}}{a}$$

$$= \frac{1}{8a} \cdot \left\{ -2x(a^2 - x^2)^{\frac{3}{2}} + a^2 x \cdot (a^2 - x^2)^{\frac{1}{2}} + a^4 . \text{arc} \left(\sin. = \frac{x}{a} \right) \right\} + C$$

$$= \frac{1}{8a} \cdot \left\{ (a^2 - x^2)^{\frac{1}{2}} \cdot x \left[a^2 - 2(a^2 - x^2) \right] + a^4 . \text{arc} \left(\sin. = \frac{x}{a} \right) \right\} + C$$

$$= \frac{1}{8a} \cdot \left\{ (a^2 - x^2)^{\frac{1}{2}} \cdot (2x^2 - a^2) \cdot x + a^4 \cdot \text{arc} \left(\sin. = \frac{x}{a} \right) \right\} + C:$$

and the integral being taken from $x = 0$, and $y = 0$, we have $C = 0$.

$$\text{Hence } h = \frac{3}{8} \cdot \frac{(a^2 - x^2)^{\frac{1}{2}} \cdot (2x^2 - a^2) \cdot x + a^4 \cdot \text{arc} \left(\sin. = \frac{x}{a} \right)}{a^3 - (a^2 - x^2)^{\frac{3}{2}}}.$$

If we take the whole curve, that is, make $x = a$, we have

$$h = \frac{3}{8} \cdot \frac{a^4 \cdot \frac{\pi}{2}}{a^5} = \frac{3\pi}{16} \cdot a.$$

Similarly, we should obtain the following results:

Ex. 2. If PAp, fig. 110, be the common parabola,

$$AG = \frac{3}{5} AM.$$

Ex. 3. If PAp be any parabola whose equation is $y^{m+n} = a^m x^n$,

$$AG = \frac{m + 2n}{2m + 3n} AM.$$

Ex. 4. If PAp be a segment of a circle whose center is C,

$$CG = \frac{PM^3}{3 \text{ area } AMP}.$$

Ex. 5. If BAb be a semi-circle; center C; (fig. 110.)

$$CG = \frac{4}{3\pi} AC.$$

Ex. 6. If PAp be any segment of an ellipse, whose semi-axes are $CA=a$, and $CB=b$;

$$CG=\frac{a^2}{b^2}\cdot\frac{PM^3}{3 \text{ area } AMP}.$$

Ex. 7. If BAb be a semi-ellipse, with center C;

$$CG=\frac{4}{3\pi}AC.$$

Ex. 8. If PAp, fig. 112, be any segment of a hyperbola whose semi-axes are $CA=a$, $CB=b$;

$$CG=\frac{a^2}{b^2}\cdot\frac{PM^3}{3 \text{ area } AMP}.$$

Ex. 9. If $P'Bbp'$ be a segment of the area contained between the two hyperbolas which are conjugate to PAp;

$$CG'-\frac{a^2}{b^2}\cdot\frac{P'M'^3-CB^3}{3P'Bbp'}.$$

Ex. 10. If BP' be a rectangular hyperbola whose asymptotes are CE and Ce, and if we complete the parallelogram $P'O$, we have for the area $PP'Q'Q$;

$$Cg=\frac{\text{area } P'O}{\text{area } P'Q'}\cdot mn.$$

Ex. 11. If PAp, fig. 110, be a cycloid; axis AC; we have for the whole cycloid,

$$AG=\frac{7}{12}AC.$$

Ex. 12. If $PApC$ be a sector of a circle; center C;

$$CG=\frac{2}{3}\cdot\frac{AC\,.\,Pp}{\text{arc } Pp}=\frac{2}{3}\cdot\frac{\text{rad. chord}}{\text{arc}}.$$

2. *A Curvilinear Area not symmetrical.*

35. Let $BCQP$, fig. 111, be a curvilinear area bounded by two curves BP, CQ, and their ordinates. Let G be its

center of gravity, and GK, GH the co-ordinates of this point parallel to the known lines Ax and Ay. Then we may find GH as before, by the formula

$$h = \frac{\int_x x \frac{dm}{dx}}{m},$$

where the value of $\frac{dm}{dx}$ is the differential coefficient of the area $BCQP$.

If $AM = x$, $MP = y$, $MQ = y'$, we have $\frac{dm}{dx} = y - y'$;

$$\therefore\ h = \frac{\int_x (y - y') x}{\int_x (y - y')} \ldots\ldots\ldots\ldots\ldots (1).$$

But if we take the integral of 1 with respect to y, supposing the integral to begin when y is y', it will be $y - y'$: or $y - y' = \int_y 1$. Hence we have for h,

$$h = \frac{\int_x x \int_y 1}{\int_x \int_y 1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

When an integral is to be found by two integrations, thus indicated by the double sign $\int_x \int_y$, the first integration is to be performed considering y as the variable quantity, and x as constant. We must then substitute for y the value which it has as a function of x, according to the manner in which the integral is limited, and must integrate the resulting expression considering x as the variable quantity.

Now if we perform the integrations with respect to y and x in a reverse order, we shall evidently obtain the same result.

Hence, instead of $\int_x x \int_y 1$, we may use $\int_y \int_x x$; and $\int_y \int_x 1$ instead of $\int_x \int_y 1$.

$$\text{But } \int_y \int_x x = \int_y \frac{x^2}{2} + C$$

$$= \int_y \frac{x^2 - x'^2}{2}$$

where x' is the value of x when the area begins, and x the

value where it ends, corresponding to a constant value of y. Hence

$$h = \frac{\int_y (x^2 - x'^2)}{2\int_y\int_x 1} \text{.........}(3).$$

In the same manner if k be the ordinate GK, we shall have

$$k = \frac{\int_x (y^2 - y'^2)}{2\int_x\int_y 1} \text{.........}(4).$$

where y' and y are the first and last values of the ordinate corresponding to a given value of x.

The value of k may be found by formulæ corresponding to any of those which we have obtained for h: and by taking a value of h and a value of k, we determine the position of the center of gravity.

Thus we may take formulæ (1) and (4), putting $\int_x (y - y')$ for $\int_x\int_y 1$.

$$\left.\begin{aligned} h &= \frac{\int_x (y - y')x}{\int_x (y - y')} \\ k &= \frac{\int_x (y^2 - y'^2)}{2\int_x (y - y')} \end{aligned}\right\} \text{.........}(5).$$

In these formulæ the value of y in terms of x is to be substituted, after which the integration is to be performed with respect to x, and to be taken between the limits corresponding to the extremities of the curve.

If the curvilinear space be bounded, as ADB, fig. 113, by the abscissa, the ordinate and the curve, we shall have $y' = 0$.

$$h = \frac{\int_x yx}{\int_x y}; \quad k = \frac{\int_x y^2}{2\int_x y} \text{.........}(6).$$

If the curvilinear space be bounded, as ACD, fig. 114, by the lines Ax, Ay, and by the curve, we might use the formulæ

$$h = \frac{\int_x xy}{\int_x y}; \quad k = \frac{\int_y yx}{\int_y x} \text{.........}(7).$$

the integration in the first beginning when $x = 0$, and in the second when $y = 0$.

Ex. 13. Let AB, fig. 113, be a parabola whose axis is AD: to find the center of gravity of the space ADB.

Let $y^2 = cx$; and by formula (6),

$$h = \frac{\int_x yx}{\int_x y}; \quad k = \frac{\int_x y^2}{2\int_x y}.$$

$$\text{Now } \int_x y = \int_x c^{\frac{1}{2}} x^{\frac{1}{2}} = \tfrac{2}{3} c^{\frac{1}{2}} x^{\frac{3}{2}},$$

$$\int_x yx = \int_x c^{\frac{1}{2}} x^{\frac{3}{2}} = \tfrac{2}{5} c^{\frac{1}{2}} x^{\frac{5}{2}},$$

$$\int_x y^2 = \int_x cx \quad = \tfrac{1}{2} c x^2;$$

$$\therefore\ h = \tfrac{3}{5} x,\ k = \tfrac{3}{8} c^{\frac{1}{2}} x^{\frac{1}{2}} = \tfrac{3}{8} y.$$

Hence, if we take $AH = \frac{3}{5} AD$, and $AK = \frac{3}{8} AC$, by completing the parallelogram, we have G the center of gravity of ABD.

3. *A Solid of Revolution.*

36. In a solid of revolution, whose axis is Ax, $\dfrac{dm}{dx} = \pi y^2$; hence

$$h = \frac{\int_x \pi y^2 x}{\int_x \pi y^2} = \frac{\int_x y^2 x}{\int_x y^2}.$$

And the center of gravity will be in the axis Ax: hence if we measure this value of h along Ax, we have the center of gravity.

Ex. 14. Let PAp, fig. 110, be a segment of a sphere whose center is C; $AC = a$, $AM = x$, $AG = h$,

$$h = \frac{8ax - 3x^2}{4(3a - x)}.$$

And for the whole hemisphere, when $x = a$,

$$h = \frac{5a}{8}.$$

Ex. 15. Let the body be a segment of a spheroid generated by the revolution of an elliptical segment PAp; the center of gravity will be the same as that of a segment of a sphere with the same axis and center C; or, as before,

$$h = \frac{8ax - 3x^2}{4(3a - x)}.$$

And for the hemispheroid, $h = \frac{5a}{8}$.

Ex. 16. Let PAp, fig. 112, be a hyperboloid with center C; $CA = a$, $AM = x$, $AG = h$;

$$h = \frac{8ax + 3x^2}{4(3a + x)}.$$

As x becomes very great, the value to which this tends is

$$h = \frac{3x}{4};$$

which agrees with the expression for a cone.

Ex. 17. If PAp be a paraboloid;

$$h = \frac{2x}{3}.$$

Ex. 18. If the figure be a frustum of a paraboloid, of which the radii of the less and of the greater ends are a and b, and the length of the axis x, the distance of the center from the smaller end, h;

$$h = \frac{a^2 + 2b^2}{a^2 + b^2} \cdot \frac{x}{3}.$$

Ex. 19. If PAp be a solid generated by the revolution of any parabola whose equation is

$$y^{m+n} = a^m x^n;$$

$$h = \frac{m + 3n}{2m + 4n} \cdot x.$$

4. *Any Solid.*

37. Let PBQ, fig. 115, represent ony solid bounded by a surface to which we have an equation in terms of three rectangular co-ordinates x, y, z. Let Ax be the direction of one of the co-ordinates, and let the body h be cut by a plane PM perpendicular to Ax. Let A be the area of the section of the body made by this plane. Then $\frac{dm}{dx}$ will $= A$.

Now the boundaries of the plane A perpendicular to AM will be determined by the co-ordinates y and z, which are perpendicular to AM, and in the plane A. Hence we shall have

$$A = \int_y z, \text{ or as before } A = \int_y\int_z 1. \quad \text{And } \frac{dm}{dx} = \int_y\int_z 1;$$

$$\therefore h = \frac{\int_x x\int_y\int_z 1}{\int_x\int_y\int_z 1}.$$

Or, since, as in Art. 35, the order of the integrations is indifferent*,

$$h = \frac{\int_x\int_y\int_z x}{\int_x\int_y\int_z 1};$$

$$\text{similarly, } k = \frac{\int_x\int_y\int_z y}{\int_x\int_y\int_z 1},$$

$$l = \frac{\int_x\int_y\int_z z}{\int_x\int_y\int_z 1}:$$

k, l being the co-ordinates of the center of gravity parallel respectively to y and to z.

If we suppose the integration in z to be performed, we shall have

* The expression for the solidity of a body is $\int_x\int_y z$. Similarly it is $\int_y\int_z x$, and $\int_z\int_x y$. The expression $\int_x\int_y\int_z 1$ comprehends all these three. For the order of the integrations is indifferent as in p. 46; and if we make the first integration with respect to z, we obtain $\int_x\int_y z$: if with respect to x, we have $\int_y\int_z x$; if with respect to y, we have $\int_z\int_x 1$. Similarly $\int_x\int_y\int_z 1$ is the same as $\int_x\int_y\int_z x$; and so of the rest.

$$h = \frac{\int_x \int_y xz}{\int_x \int_y z}.$$

$$k = \frac{\int_x \int_y yz}{\int_x \int_y z},$$

$$l = \frac{\int_x \int_y z^2}{2\int_x \int_y z}.$$

And z being known in terms of x, y, its value may be substituted, and the integrations in y performed, between the proper limits. Then the value of y in terms of x may be substituted; and the integrations performed with respect to x will give the value of h.

Ex. 20. Let the body be a fourth part of a paraboloid of revolution; as $ABCD$, fig. 115; cut off by a plane BAC perpendicular to the axis, and by two planes BAD, CAD, perpendicular to the preceding and to each other; to find its center of gravity.

Let A be the origin, and AB, AC, AD the axes of the rectangular co-ordinates x, y, z, respectively. If $AM = x$, $AN = MO = y$, $OP = z$, the equation to the surface will be

$$x^2 + y^2 + bz = a^2,$$

where AB or $AC = a$, and the axis $AD = \frac{a^2}{b}$,

$$\int_x \int_y z = \int_x \int_y \left(\frac{a^2 - x^2 - y^2}{b}\right) = \frac{1}{b} \int_x \int_y (a^2 - x^2 - y^2).$$

And, integrating first for y,

$$= \frac{1}{b} \int_x \left(a^2 y - x^2 y - \frac{y^3}{3}\right).$$

The limits of the integration for y are determined by the nature of the part considered; if it is to be bounded by a plane RNO parallel to the plane of xz at a distance AN, y must be taken from o to AN; and in the next integration must be supposed constant. Hence we have

$$\int_x \int_y z = \frac{1}{b}\left(a^2xy - \frac{x^3y}{3} - \frac{y^3x}{3}\right).$$

Where the limits of the integration for x are determined in the same way as for y. If the solid be bounded by a plane QMO parallel to the plane of yz, at a distance $x = AM$, the quantity now found expresses the solid; or

$$\text{solid } AP = \frac{xy}{b}\left(a^2 - \frac{x^2+y^2}{3}\right) = \frac{xy.(3a^2 - x^2 - y^2)}{3b}.$$

If the solid be not bounded by a plane RNO, but continued till its surface meets the plane CAB in Cm, we must, after the integration for y, put for y the value which it assumes by making

$$z = 0, \text{ or } x^2 + y^2 = a^2, \text{ whence } y^2 = a^2 - x^2.$$

Hence

$$\int_x \int_y z = \frac{1}{b}\int dx\left(a^2y - x^2y - \frac{y^3}{3}\right)$$

$$= \frac{1}{b}\int_x\left(a^2 - x^2 - \frac{y^2}{3}\right).y$$

$$= \frac{2}{3b}\int_x (a^2 - x^2)^{\frac{3}{2}},$$

which (taken from $x = 0$) gives the solid $ACmQD$

$$= \frac{2}{3.8.b}\left\{2x(a^2 - x^2)^{\frac{3}{2}} + 3a^2x(a^2 - x^2)^{\frac{1}{2}} + 3a^4.\text{arc}\left(\sin. = \frac{x}{a}\right)\right\}.$$

And if we take the whole solid, x must be taken $= a$; in this case the arc will become $\frac{\pi}{2}$, and we shall have

$$\text{whole solid } ABCD = \frac{2}{3.8.b}.3a^4.\frac{\pi}{2} = \frac{\pi a^4}{8b}.$$

The solidity of the whole might be more simply found; for it will manifestly be $\frac{1}{4}$ of the whole paraboloid; and a

paraboloid is $\frac{1}{2}$ the cylinder on the same base (rad. $= a$), and with the same altitude; hence

$$\text{whole solid} = \frac{1}{4}\cdot\frac{1}{2}\pi a^2 . \frac{a^2}{b} = \frac{\pi a^4}{8b}, \text{ as before.}$$

We now proceed to find $\int_x \int_y x z$,

$$\int_x \int_y x z = \int_x \int_y x \left(\frac{a^2 - x^2 - y^2}{b}\right)$$

$$= \frac{1}{b}\int_x x \int_y (a^2 - x^2 - y^2)$$

$$= \frac{1}{b}\int_x x \left(a^2 y - x^2 y - \frac{y^3}{3}\right);$$

the limits of y as before. On the first supposition, that the body is bounded by planes RNO, QMO, we have

$$\int_x \int_y x z = \frac{1}{b}\left(\frac{a^2 x^2 y}{2} - \frac{x^4 y}{4} - \frac{x^2 y^3}{6}\right)$$

$$= \frac{x^2 y (6a^2 - 3x^2 - 2y^2)}{12b};$$

hence, for AP,

$$h = \frac{b\,\{x^2 y (6a^2 - 3x^2 - 2y^2)\}}{4b\,\{xy\,(3a^2 - x^2 - y^2)\}}$$

$$= \frac{x}{4}\cdot\frac{6a^2 - 3x^2 - 2y^2}{3a^2 - x^2 - y^2}.$$

If we suppose $AMON$, the base of the figure, to be a square, or $y = x$; this becomes

$$h = \frac{x}{4}\cdot\frac{6a^2 - 5x^2}{3a^2 - 2x^2}.$$

If we suppose the quadrilateral curve surface $DQPR$ to have its angle P in the circumference of the base BC, as at m, we shall have $z = 0$; and hence

$x^2 + y^2 = a^2$, or $y^2 = a^2 - x^2$; and hence for the solid Mr,

$$h = \frac{x}{4} \cdot \frac{4a^2 - x^2}{2a^2}.$$

On the second supposition, that the surface of the solid is to be continued till it meets the plane ABC, we must, after the integration for y, substitute for y its value in that plane, that is, $y = \sqrt{(a^2 - x^2)}$; hence we have

$$\int_x \int_y xz = \frac{1}{b} \int x \, dx \left(a^2 y - x^2 y - \frac{y^3}{3}\right)$$

$$= \frac{1}{b} \int_x x \left(a^2 - x^2 - \frac{a^2 - x^2}{3}\right) . \sqrt{(a^2 - x^2)}$$

$$= \frac{2}{3b} . \int_x x . (a^2 - x^2)^{\frac{3}{2}}$$

$$= \frac{2}{3b} . \left(\frac{a^5}{5} - \frac{(a^2 - x^2)^{\frac{5}{2}}}{5}\right),$$

taking the integral from $x = 0$; and for the whole solid $ABCD$, or when $x = 0$, it becomes

$$= \frac{2}{3b} . \frac{a^5}{5} = \frac{2a^5}{15b}.$$

Hence, for the whole solid $ABCD$,

$$h = \frac{2a^5}{15b} . \frac{8b}{\pi a^4} = \frac{16a}{15\pi} = \frac{a}{3}, \text{ nearly.}$$

In the same way it might be shewn that, for the part of the solid bounded by planes parallel to the planes of xz and xy, we have

$$k = \frac{y}{4} . \frac{6a^2 - 3y^2 - 2x^2}{3a^2 - y^2 - x^2};$$

$$l = \frac{3\left\{a^4 - \frac{2}{3}a^2(x^2 + y^2) + \frac{2}{9}x^2 y^2 + \frac{1}{5}(x^4 + y^4)\right\}}{2b . (3a^2 - x^2 - y^2)}.$$

And for the whole solid,

$$k = \frac{16a}{15\pi},$$

$$l = \frac{b^2}{3a} = \frac{\text{axis}}{3},$$

which last result also follows from Ex. 17.

5. *A Plane Curve.*

38. When the body is a curve lying in one plane, if we suppose it to be a physical line of inconsiderable thickness, ds being the differential of its length, $\frac{dm}{dx}$ will be as $\frac{ds}{dx}$.

Hence,

$$h = \frac{\int_x x \frac{ds}{dx}}{\int_x \frac{ds}{dx}}.$$

But we have $\frac{ds}{dx} = \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$ (Lacroix, *Elementary Treatise*, Art. 75.). Therefore

$$h = \frac{\int_x x \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\int_x \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}.$$

Similarly, $$k = \frac{\int_x y \frac{ds}{dx}}{\int_x \frac{ds}{dx}};$$

or, $$k = \frac{\int_x y \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\int_x \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}.$$

If the curve be symmetrical with respect to Ax, it will be sufficient to find h, since the center of gravity will be in Ax.

Ex. 21. Let PAp, fig. 110, be a circular arc, center C, radius $= a$.

Let arc $AP = s$; $\therefore\ CM = x = a \cdot \cos. \frac{s}{a}$;

$$\therefore\ \int_s x \frac{ds}{dx} = a \int_s \cos. \frac{s}{a} \cdot \frac{ds}{dx} = a^2 \sin. \frac{s}{a}.$$

And if the whole arc be $2l$, and its middle point A, the integral must be taken from

$$s = -l \text{ to } s = l; \ \therefore\ \int_s x \frac{ds}{dx} = 2a^2 . \sin. \frac{l}{a}.$$

$$\text{Hence } CG = h = \frac{2a^2 . \sin. \frac{l}{a}}{2l} = \frac{a . 2 \sin. l \, (\text{rad.} = a)}{2l}$$

$$= \frac{\text{radius . chord}}{\text{arc}}.$$

Cor. Hence for the semi-circle, $h = \frac{2a}{\pi}$.

Ex. 22. Let APB, fig. 113, be a semi-cycloid with axis AD.

$$AH = h = \frac{AD}{3}, \ HG = k = DB - \frac{2}{3} AD.$$

Cor. Hence $CK = DH$.

Ex. 23. Let PAp, fig. 110, be a catenary of which A is the lowest point. Take AD vertical and equal to a length of the string equivalent to the tension; then

$$DG = \tfrac{1}{2} DM + \frac{DA . MP}{PAp}.$$

6. *A Curve of double Curvature.*

39. Let s be the length of the curve; and, as before, $\frac{dm}{dx} = \frac{ds}{dx}$. Then, if x, y, z be the rectangular co-ordinates to the curve,

$$\frac{ds}{dx} = \sqrt{\left(1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}\right)}, \text{ and}$$

$$h = \frac{\int_x x \frac{ds}{dx}}{\int_x \frac{ds}{dx}}; \quad k = \frac{\int_x y \frac{ds}{dx}}{\int_x \frac{ds}{dx}}, \quad l = \frac{\int_x z \frac{ds}{dx}}{\int_x \frac{ds}{dx}}.$$

Ex. 24. Let the curve be the thread of a screw, of which the axis is Az.

This thread, projected on the plane xy, will become a circle; and if a be the radius of this circle, $a^2 = x^2 + y^2$. Also $\frac{x}{a}$ will be the cosine of the arc of this circle, corresponding to any point in the curve, and z will be proportional to this arc. Hence the equations to the curve are

$$y = \sqrt{(a^2 - x^2)},$$

$$z = m \,.\, \text{arc}\left(\cos. = \frac{x}{a}\right),$$

m being a constant quantity, and the thread of the screw being supposed to begin in the line Ax.

$$\text{Hence } \frac{dy}{dx} = -\frac{x}{\sqrt{(a^2 - x^2)}},$$

$$\frac{dz}{dx} = -\frac{m}{\sqrt{(a^2 - x^2)}};$$

$$\therefore \frac{ds}{dx} = \sqrt{\left(1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}\right)}$$

$$= -a\frac{\sqrt{(a^2+m^2)}}{\sqrt{(a^2-x^2)}}.$$

Hence also $\dfrac{ds}{dx} = \dfrac{\sqrt{(a^2+m^2)}}{m}\dfrac{dz}{dx}$,

and $s = \dfrac{\sqrt{(a^2+m^2)}}{m}z.$

Now $\int_x x\dfrac{ds}{dx} = \int_x -x\dfrac{\sqrt{(a^2+m^2)}}{\sqrt{(a^2-x^2)}}$

$$= \sqrt{(a^2+m^2)}\cdot\sqrt{(a^2-x^2)};$$

which begins when $x = a$.

Also $\int_x \dfrac{ds}{dx} = \int_x -\sqrt{(a^2+m^2)}$

$$= (a-x)\sqrt{(a^2+m^2)},$$

which also begins when $x = a$.

And $\int_x z\dfrac{ds}{dx} = \int_x \dfrac{\sqrt{(a^2+m^2)}}{m}z\dfrac{dz}{dx}$

$$= \frac{\sqrt{(a^2+m^2)}}{m}\cdot\frac{z^2}{2}.$$

Hence

$$h = \frac{m\sqrt{(a^2-x^2)}}{z};$$

$$k = \frac{m(a-x)}{z};$$

$$l = \frac{z}{2}.$$

If $x = a$, that is, if the spiral consist of a complete number of revolutions, $h = 0$, $k = 0$. In this case the center of gravity is in the axis, and in the middle of its length.

If $x = 0$, $h = \frac{ma}{z}$, $k = \frac{ma}{z}$: in this case the spiral consists of a complete number of revolutions together with a quarter of a revolution.

7. *A Surface of Revolution.*

40. If s be the length of the curve, $2\pi y \frac{ds}{dx}$ is the differential coefficient of the surface with respect to x, and, as before, this may be put for $\frac{dm}{dx}$. Also the center of gravity will be in the axis of revolution. Hence

$$h = \frac{\int_x xy \frac{ds}{dx}}{\int_x y \frac{ds}{dx}} = \frac{\int_x yx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\int_x y \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}.$$

Ex. 25. If the surface be a cone, and h the distance from the vertex,

$$h = \frac{2x}{3}.$$

Ex. 26. If the surface be a sphere, and h measured from the vertex,

$$h = \frac{x}{2}.$$

8. *Any Surface.*

41. Let the surface be defined by any equation $u = 0$, between x, y, z; whence we may find z in terms of x and y.

Let $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$; h, k, l as in Art. 37. The differ-

ential coefficient, taken with regard to x and y successively, of the surface, is $(1+p^2+q^2)^{\frac{1}{2}}$; hence as before,

$$h = \frac{\int_x \int_y x\,(1+p^2+q^2)^{\frac{1}{2}}}{\int_x \int_y (1+p^2+q^2)^{\frac{1}{2}}};$$

$$k = \frac{\int_x \int_y y\,(1+p^2+q^2)^{\frac{1}{2}}}{\int_x \int_y (1+p^2+q^2)^{\frac{1}{2}}};$$

$$l = \frac{\int_x \int_y z\,(1+p^2+q^2)^{\frac{1}{2}}}{\int_x \int_y (1+p^2+q^2)^{\frac{1}{2}}}.$$

Ex. 27. Let a conical surface be divided into four parts by two planes perpendicular to each other, passing through the axis. To find the center of gravity of one of these parts: as *BCD*, fig. 115; *DB* and *DC* being supposed to be straight lines.

If we make the vertex *D* the origin of co-ordinates, the axis the line of z, and measure z, y, parallel to *AB*, *AC*, respectively, we have

$$z = m\sqrt{(x^2+y^2)};$$

Where m is the tangent of the angle which the slant side makes with the base.

Hence

$$p = \frac{dz}{dx} = \frac{mx}{\sqrt{(x^2+y^2)}},$$

$$q = \frac{dz}{dx} = \frac{my}{\sqrt{(x^2+y^2)}};$$

$$\therefore (1+p^2+q^2)^{\frac{1}{2}} = (1+m^2)^{\frac{1}{2}},$$

$$\therefore \int_x \int_y (1+p^2+q^2)^{\frac{1}{2}} = (1+m^2)^{\frac{1}{2}} \cdot \int_x y.$$

And if the curve surface be a quadrilateral figure *DQPR* bounded by planes parallel to those of the co-ordinates,

$$\text{this} = (1+m^2)^{\frac{1}{2}}\, xy.$$

But if we take the surface as bounded by a plane BAC perpendicular to the axis at the distance $a = DA$, we must have, after the integration for y, $x = a$, the axis;

$$\text{Now } a^2 - m^2 x^2 = m^2 y^2,\ y = \frac{\sqrt{(a^2 - m^2 x^2)}}{m},$$

$$\therefore \int_x \int_y (1 + p^2 + q^2)^{\frac{1}{2}} = \frac{(1 + m^2)^{\frac{1}{2}}}{m} \cdot \int_x (a^2 - m^2 x^2)^{\frac{1}{2}}$$

$$= \frac{(1 + m^2)^{\frac{1}{2}}}{2m} \left\{ (a^2 - m^2 x^2)^{\frac{1}{2}} \cdot x + \frac{a^2}{m} \cdot \text{arc} \left(\sin. = \frac{mx}{a} \right) \right\} + \text{const.}$$

and, from $x = 0$ to $x = AB = \dfrac{a}{m}$

$$= \frac{(1 + m^2)^{\frac{1}{2}}}{2m} \cdot \frac{a^2 \cdot \pi}{2m} = \frac{(1 + m^2)^{\frac{1}{2}} \cdot \pi a^2}{4m^2},$$

which might be deduced also from the known method of finding the surface of a cone.

To find the numerator of h, we have it

$$= (1 + m^2)^{\frac{1}{2}} \int_x \int_y x = (1 + m^2)^{\frac{1}{2}} \int_x y\, x,$$

and, for the quadrilateral $DPQR$, $= \dfrac{(1 + m^2)^{\frac{1}{2}}}{2}\, y x^2$.

But, for the whole surface DBC,

$$= \frac{(1 + m^2)^{\frac{1}{2}}}{m} \int_x (a^2 - m^2 x^2)^{\frac{1}{2}} x$$

$$= -\frac{(1 + m^2)^{\frac{1}{2}}}{3m^3} (a^2 - m^2 x^2)^{\frac{3}{2}} + \text{constant}:$$

and, taken from $x = 0$ to $x = AB = \dfrac{a}{m}$

$$= \frac{(1 + m^2)^{\frac{1}{2}} a^3}{3m^3}.$$

The numerator of k will manifestly be the same.

Similarly, for the numerator of l,

$$\int_x \int_y (1+p^2+q^2)^{\frac{1}{2}} z = (1+m^2)^{\frac{1}{2}} \int_x \int_y z$$

$$= (1+m^2)^{\frac{1}{2}} \int_x \int_y m\,(x^2+y^2)^{\frac{1}{2}}$$

$$= (1+m^2)^{\frac{1}{2}} \,.\, m \,.\, \int_x \left\{ \frac{(x^2+y^2)^{\frac{1}{2}} y}{2} + \frac{x^2}{2} \,\mathrm{l}\, \frac{y+\sqrt{(x^2+y^2)}}{x} \right\},$$

in which the denominator x is given to the quantity under the logarithmic sign that the integral may begin when $y=0$. For the quadrilateral surface $DPQR$, we must now integrate for x, supposing y constant; and the double integral becomes

$$= \frac{(1+m^2)^{\frac{1}{2}} \,.\, m}{6} \left\{ 2xy \,.\, (x^2+y^2)^{\frac{1}{2}} + x^3 \,\mathrm{l}\, \frac{y+\sqrt{(x^2+y^2)}}{x} \right.$$

$$\left. + y^3 \,\mathrm{l}\, \frac{x+\sqrt{(x^2+y^2)}}{y} \right\}.$$

For the whole surface DBC, we must, after integrating for y, put for y the value $\dfrac{\sqrt{(a^2-m^2x^2)}}{m}$; and it becomes,

$$= \frac{(2+m^2)^{\frac{1}{2}} \,.\, m}{2} \int_x \left\{ \frac{a\sqrt{(a^2-m^2x^2)}}{m^2} + x^2 \,\mathrm{l}\, \frac{\sqrt{(a^2-m^2x^2)}+a}{mx} \right\};$$

and the integral being taken from $x=0$ to $x=\dfrac{a}{m}$; this becomes

$$= \frac{(1+m^2)^{\frac{1}{2}} \,.\, \pi a^3}{6m^2}.$$

Hence, for the whole conical surface,

$$h = k = \frac{(1+m^2)^{\frac{1}{2}} \,.\, a^3}{3m^3} \cdot \frac{4m^2}{(1+m^2)^{\frac{1}{2}} \,.\, \pi a^2} = \frac{4a}{3\pi m},$$

$$l = \frac{(1+m^2)^{\frac{1}{2}} \,.\, \pi a^3}{6m^2} \cdot \frac{4m^2}{(1+m^2)^{\frac{1}{2}} \,.\, \pi a^2} = \frac{2a}{3},$$

which last agrees with Ex. 33, as it should.

For the center of gravity of the quadrilateral surface $DPQB$, where $AM=x$, $AN=y$, we have for h, k, l, other expressions which are easily deduced from the results given above.

9. *Guldinus's Properties**.

42. Prop. *If any plane figure revolve about an axis in its own plane, the content of the solid generated by this figure in its revolution is equal to a prism whose base is the revolving figure, and its height the length of the path described by the center of gravity of the plane figure.*

The figure may either be bounded by straight lines, or curves; or by a combination of the two; and the revolution may take place either through a whole circumference or any part of it.

We shall suppose the whole of the revolving figure to be on one side of the axis.

Let AB, fig. 116, be the axis of revolution, PQR the figure; G its center of gravity; GK, PQM, ordinates perpendicular to AB. And let the figure revolve into the position $P'Q'R'$; the angle PMP' being $=\theta$. Also let $AM = x$, $PM = y$, $MQ = y'$, $GK = k$.

The sector $PMP' = \frac{1}{2}y^2\theta$, and $QMQ' = \frac{1}{2}y'^2\theta$. Hence

$$PQQ'P' = \frac{1}{2}(y^2 - y'^2)\,\theta;$$

and hence the small increment of the solid PR' corresponding to δx is $\frac{1}{2}(y^2 - y'^2)\,\theta\,\delta x$. Hence the solid

$$= \frac{\theta}{2} \textstyle\int_x (y^2 - y'^2).$$

Also the prism whose base is PQR, and altitude the arc GG', is $= PQR \,.\, GG'$; and area $PQR = \int_x (y - y')$, $GG' = k\theta$. Hence this prism $= k\theta \int_x (y - y')$.

But by formula (5) for k, in Art. 35, we have

$$k \textstyle\int_x (y - y') = \frac{1}{2} \int_x (y^2 - y'^2).$$

* The propositions known by this name were discovered by Pappus, and republished about 1640, by Guldin or Guldinus, a Jesuit, who was Professor of Mathematics at Rome.

Therefore the figure described by the revolution of PQR is equal to the prism mentioned in the Proposition.

If the figure be composed of several curves, or of straight lines, both the numerator and denominator of k will consist of several integrals added together, corresponding to the different parts of the figure. Also both the area of the figure, and the content of the solid will consist of parts corresponding to these; and the solid and the prism will be found to be equal in the same manner as before.

43. Prop. *If any plane figure revolve about any axis in its own plane, the area of the surface generated by the perimeter of this figure in its revolution is equal to a rectangle, one of whose sides is the perimeter, and the other the length of the path described by the center of gravity of the perimeter.*

The denominations remaining as in last Article, let δs be the small increment of the length of the curve corresponding to δx; and since $y\theta$ is the length of the path described by P, $y\theta . \delta s$ is the increment of the surface described by the revolution; and $\theta \int_x y \dfrac{ds}{dx}$ is the whole surface.

Also the whole perimeter is $\displaystyle\int_x \frac{ds}{dx}$; and if G be now its center of gravity of the perimeter, $k\theta$ is the path described by the center of gravity of the perimeter; and $k\theta \displaystyle\int_x \frac{ds}{dx}$ is the rectangle mentioned in the proposition.

But by Art. 38,

$$k \int_x \frac{ds}{dx} = \int_x y \frac{ds}{dx},$$

whence the proposition is manifest.

44. Hence we may find the contents and areas of surfaces of revolution whenever we can find the area or perimeter of the revolving figure and its center of gravity.

Ex. 28. Let the figure be a circle which, revolving round an axis without it, generates a solid, resembling a cylinder

bent so as to return into itself, or the ring of an anchor. The center of the circle will be the center of gravity both of the area and of the perimeter. Hence, by Article 42, the solid content of such a ring is equal to the cylinder whose base is the revolving circle and its length the circle described by the center of the circle. Also by Article 43, the surface of the ring is equal to the rectangle contained by the circumference of the revolving circle and the path of its center; that is, it is equal to the surface of the cylinder before-mentioned. Hence if we could suppose that the ring was cut through in some part, and unrolled into a cylinder so that its axis should remain of the same length as before, both the solidity and the surface would continue unaltered.

CHAP. V.

THE EQUILIBRIUM OF A FLEXIBLE BODY.

45. The equilibrium of a flexible body depends upon the same conditions as that of a rigid one, and may be deduced from the principles already laid down. These principles may be applied by observing that, in all cases, *a flexible body may be supposed to become rigid after the equilibrium is established:* or, that the forces which keep a flexible body at rest would keep at rest a rigid body of the same form. For after the body has assumed the form which the forces produce in it, there is no tendency to change the form; hence it makes no difference whether we suppose the body to have the property of resisting a change of form or not: that is, it makes no difference whether we suppose the body to be rigid or flexible.

We shall suppose the bodies to be *perfectly* flexible, that is, to offer *no* resistance to any change of figure.

The *Tension* of a string or chain is the force exerted by one part upon another contiguous part in the direction of its length. Every point of the string must be acted upon by equal and opposite forces of this kind: and a force of the same kind is exerted upon any fixed point to which the string is attached.

We shall consider the equilibrium of a flexible *line**, acted on by various forces. This line may be supposed to be a *cord*, indefinitely slender and perfectly void of stiffness; or a *chain* composed of indefinitely small links. On this account the curve formed by the line is called the *Catenary*.

1. *The Catenary, when a uniform Chain is acted on by Gravity.*

46. Prop. *To find the equation to the catenary between* z *and* s, *beginning at the lowest point.*

Let AB, fig. 117, represent the catenary. Let C be the lowest point, CM vertical $= x$, MP horizontal $= y$, $CP = s$. The portion CP may be supposed to become rigid after it has assumed the form of equilibrium; and since its weight and figure remain the same as before, it will be supported in the same manner. Now the forces which act upon the portion CP are, besides its own gravity, the tension at C and the tension at P: and these three forces must keep CP in equilibrium. Also the tensions are in the directions of the tangents RC and RP at C and P.

Let PR meet MC in T, PM will be parallel to RC, and hence the three lines MT, PM, TP are parallel to the directions of the three forces (gravity, tension at C, tension at P), which keep CP at rest, and hence (*Elem. Tr.*) the forces will be as those three lines. Hence

$$\frac{\text{tension at } C}{\text{weight of } CP} = \frac{MP}{TM}.$$

* Flexible bodies may be lines, surfaces, or solids. A flexible *line* can in all cases be extended into a straight line. A flexible surface is not necessarily susceptible of being unrolled into a plane, without stretching or tearing; if it be capable of this, it is called a *developable surface*.

Let the tension of the string at C be equal to the weight of a length c of the string; the weight of the length CP will be as CP or s; and the first side of the above equation will be $\frac{c}{s}$. Also $\frac{dy}{dx}$ will be equal to the second side.

$$\text{Hence } \frac{c}{s} = \frac{dy}{dx} \text{ .... (1);}$$

from which equation the properties of the curve may be deduced.

If we square both sides of equation (1) and add unity to them, we have

$$\frac{c^2 + s^2}{s^2} = 1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2;$$

$$\therefore \quad \frac{dx}{ds} = \frac{s}{\sqrt{(c^2 + s^2)}} \text{ ...... (2),}$$

and integrating with respect to s, supposing $s = 0$ when $x = 0$,

$$x + c = \sqrt{(c^2 + s^2)} \text{ .... (3).}$$

Hence also we find

$$s = \sqrt{(x^2 + 2cx)} \text{ ...... (4).}$$

COR. 1. If the angle which the curve makes with the vertical be called a, we have

$$\tan. a = \frac{dy}{dx} = \frac{c}{s};$$

$$\cos. a = \frac{dx}{ds} = \frac{s}{\sqrt{(c^2 + s^2)}}.$$

COR. 2. For the tension at any point P,

$$\frac{\text{tension at } P}{\text{weight of } CP} = \frac{TP}{TM} = \frac{ds}{dx} = \frac{\sqrt{(c^2 + s^2)}}{s};$$

and weight of $CP = s$; $\therefore$ tension at $P = \sqrt{(c^2 + s^2)} = x + c$.

Hence, and from the last Corollary, it appears that

$$\text{tension at } P = \frac{s}{\cos. a}.$$

Cor. 3. If we take $CD = c$, and draw DQ horizontal, and PQ vertical, $PQ = DM = x + c =$ tension at P.

Cor. 4. If we put $DM = u$, $x = u - a$, and hence

$$s = \sqrt{(u^2 - a^2)},$$

$$u = \sqrt{(s^2 + a^2)}.$$

47. Prop. *To find the equation between* y *and* s.

As in last Article,

$$\frac{s}{c} = \frac{dx}{dy} \dots\dots\dots\dots\dots\dots\dots (1)$$

$$\frac{s^2 + c^2}{c^2} = 1 + \left(\frac{dx}{dy}\right)^2 = \left(\frac{ds}{dy}\right)^2;$$

$$\frac{1}{c} \cdot \frac{dy}{ds} = \frac{1}{\sqrt{(s^2 + c^2)}} \dots\dots\dots\dots (2).$$

Integrating with respect to s, supposing $y = 0$ when $s = 0$,

$$\frac{y}{c} = \text{l}\,\frac{s + \sqrt{(s^2 + c^2)}}{c} \dots\dots (3).$$

Hence, $\epsilon^{\frac{y}{c}} = \dfrac{s + \sqrt{(s^2 + c^2)}}{c}$,

$$\therefore \epsilon^{-\frac{y}{c}} = \frac{c}{\sqrt{(s^2 + c^2)} + s} = \frac{\sqrt{(s^2 + c^2)} - s}{c}.$$

Subtracting, and reducing

$$s = \frac{c}{2}\left(\epsilon^{\frac{y}{c}} - \epsilon^{\frac{y}{c}}\right) \dots\dots\dots\dots (4).$$

Cor. We have

$$\frac{\text{tension at } P}{\text{tension at } C} = \frac{TP}{TM} = \frac{ds}{dy} = \frac{\sqrt{(s^2 + c^2)}}{c}.$$

Also as before, tension at $P = \dfrac{c}{\sin. \alpha}$.

48. Prop. *To find the equation between* x *and* y.

If in the equation (1) of last Article we put the value of s from (4), we have

$$\frac{dx}{dy} = \frac{1}{2}\left(\epsilon^{\frac{y}{c}} - \epsilon^{-\frac{y}{c}}\right) \ldots\ldots\ldots\ldots (1).$$

Integrate with respect to y ($x = 0$ when $y = 0$),

$$x + c = \frac{c}{2}\left(\epsilon^{\frac{y}{c}} + \epsilon^{-\frac{y}{c}}\right) \ldots\ldots\ldots\ldots\ldots (2).$$

Again, if in equation (1) of Art. 46, we put for s its value from (4) of that Article, we have

$$\frac{dy}{dx} = \frac{c}{\sqrt{(x^2 + 2cx)}} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Integrate with respect to x,

$$\therefore y = c\,\mathrm{l}\,\frac{x + c + \sqrt{(x^2 + 2cx)}}{c} \ldots\ldots (4).$$

49. Prop. *To find the equations to the catenary beginning from any point.*

Let A, fig. 118, be a point which is considered as the beginning of the catenary, AP any arc. Let the curve of equilibrium be continued if necessary, and let C be its lowest point. Let AN vertical $= x$, NP horizontal $= y$, $AP = s$.

The portion AP will be kept in equilibrium in the same form whether we suppose it to be acted on at A by the tension of CA, or by the re-action of a fixed point. But if we suppose AP to be a portion of CAP, its form will be determined as in the preceding Articles. Let c be the tension at the lowest point C, $CP = s'$, and we have as before,

$$\frac{c}{s'} = \frac{dy}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

for $\dfrac{dy}{dx}$ is the same whether x and y be CM, MP, or AN, NP.

Let the tension at $A = a$, and the angle which the curve makes with the vertical $= \alpha$. Also let $CA = m$. Then by Art. 46, Cor. 2, $a \cos. \alpha = m$. Also $s' = m + s = a \cos. \alpha + s$, and $\frac{ds}{dx} = \frac{ds'}{dx}$. And by Cor. to Art. 47, $a \sin. \alpha = c$.

By equation (1),

$$\frac{c^2 + s'^2}{s'^2} = 1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{ds^2}{dx}\right) = \left(\frac{ds'^2}{dx}\right);$$

$$\therefore \frac{dx}{ds'} = \frac{s'}{\sqrt{(c^2 + s'^2)}} \text{ .... } (2),$$

$x + C = \sqrt{(c + s'^2)}$, and putting for c and s' their values, observing that $x = 0$ when $s = 0$.

$$x + a = \sqrt{(a^2 + 2as \cos. \alpha + s^2)} \text{ ....} (3).$$

Hence we find

$$s = \pm \sqrt{(x^2 + 2ax + a^2 \cos.^2 \alpha)} - a . \cos. \alpha \text{ ....} (4).$$

The double sign indicates that there are two arcs corresponding to the same value of x, as AP and AP' in the figure.

In the same manner if we take

$$\frac{s'}{c} = \frac{dx}{dy},$$

we shall find

$$\frac{1}{c}\frac{dy}{ds'} = \frac{1}{\sqrt{(c^2+s'^2)}} \text{ .... } (5),$$

$$\frac{y}{c} = \mathrm{l}\,\frac{s' + \sqrt{(c^2 + s'^2)}}{c},$$

$$y = a \sin. \alpha \,\mathrm{l}\,\frac{s + a \cos. \alpha + \sqrt{(s^2 + 2as \cos. \alpha + a^2)}}{a(1 + \cos. \alpha)} \text{ ...} (6).$$

And hence

$$a(1 - \cos. \alpha)\,\epsilon^{\frac{y}{a \sin. \alpha}} = \sqrt{(s^2 + 2as \cos. \alpha + a^2)} + s + a \cos. \alpha,$$

whence

$$a(1 - \cos. a)\,\epsilon^{-\frac{y}{a \sin. a}} = \sqrt{(s^2 + 2as \cos. a + a^2)} - (s + a \cos. a),$$

as appears by multiplying the equations. Hence, subtracting and dividing by 2,

$$s + a \cos. a = \frac{a}{2}\left\{1 + \cos. a)\,\epsilon^{\frac{y}{a \sin. a}} - (1 - \cos. a)\,\epsilon^{-\frac{y}{a \sin. a}}\right\} \ldots (7).$$

Now

$$s + a \cos. a = s' = c\frac{dx}{dy} = a \sin. a \frac{dx}{dy},$$

Hence integrating both sides with regard to x, and dividing by $a \sin. a$,

$$x + a = \frac{a}{2}\left\{(1 + \cos. a)\,\epsilon^{\frac{y}{a \sin. a}} + (1 - \cos. a)\,\epsilon^{-\frac{y}{a \sin. a}}\right\} \ldots\ldots (8)$$

And in nearly the same manner as before, we should find

$$y = a \cos. a\, \mathrm{l}\, \frac{x + a \pm \sqrt{(x^2 + 2ax + a^2 \cos.^2 a)}}{a(1 \pm \cos. a)} \ldots\ldots (9).$$

Cor. It appears that $\dfrac{dy}{dx} = \dfrac{a \sin. a}{a \cos. a + s}$.

50. By means of the formulæ thus obtained we may solve the following Problems.

Prob. I. *A chain of given length $BCB' = 2l$, fig. 118, hangs from two given points B, B' in the same horizontal line, of which the distance $BB' = 2h$ is given; to find its position.*

The middle point will here be the lowest, and the chain will form a symmetrical figure with respect to the axis CE;

$$CB = CB' = l, \quad EB = EB' = h.$$

Let a be the angle which the curve at B makes with the vertical line; and by equation (3) Art. 47,

$$\frac{y}{c} = \mathrm{l}\left(\frac{s}{c} + \frac{\sqrt{(s^2 + c^2)}}{c}\right)$$

$$= \mathrm{l}\left\{\frac{dx}{dy} + \frac{ds}{dy}\right\}$$

$$= \mathrm{l}\left\{\frac{\cos. a}{\sin. a} + \frac{1}{\sin. a}\right\}.$$

$$\text{Also } c = s\frac{dy}{dx} = l \tan. a, \text{ and } y = h;$$

$$\therefore h = l \tan. a \,\mathrm{l}\, \frac{1 + \cos. a}{\sin. a} = \tan. a \,\mathrm{l}\, \text{co-tan.} \frac{a}{2};$$

$$\therefore \frac{h}{l} = -\tan. a \,.\, \mathrm{l} \tan. \frac{a}{2}.$$

From this equation we have to determine a. This cannot be done directly, but it is easy to approximate to it with sufficient rapidity. For this purpose it will be proper to adapt the formula last found, which is calculated with Napierian logarithms and a radius $= 1$, to the common tables. Let Tan. be the tangent and R the radius of the tables; the Napierian logarithm of $10 = 2.3025851 = M$; and making log. signify the logarithm to base 10, we have

$$\frac{h}{l} = -\frac{\text{Tan.} a}{R} . M . \log. \frac{\text{Tan.} \frac{a}{2}}{R};$$

$$= -\frac{\text{Tan.} a}{R} . M . \log. \frac{R}{\text{Co-tan.} \frac{a}{2}}$$

$$= \frac{\text{Tan.} a}{R} . M . \left(\log. \text{Co-tan.} \frac{a}{2} - \log. R\right);$$

$$\therefore \log. \frac{h}{l} = \log. \text{Tan.} a + \log. \left(\log. \text{Co-tan.} \frac{a}{2} - \log. R\right) + \log. M - \log. R,$$

where $\log. R = 10$, $\log. M = .3622157$.

Assuming values of a, we may calculate $\frac{h}{l}$, and by observing the error of the result obtain a more accurate value of a.

Ex. Let the string $BCB' = 2BB'$, to find the position.

We have $\log.\frac{h}{l} = \log.\frac{1}{2} = \bar{1}.6989700.$

By a few trials we shall find that $a = 13^0$ will nearly give this value by the formula.

13^0 would give $\log.\frac{h}{l} = \bar{1}.7002484$; therefore 13^0 is too large:

$12^0.30'$............$\log.\frac{h}{l} = \bar{1}.6904752$; therefore $12^0.30'$ is too small.

Hence, since the differences of the results, when very small, are nearly proportional to the differences of the suppositions;

$$7002484 - 6904752 : 7002484 - 6989700 :: 30' : 4', \text{ nearly.}$$

Therefore $a = 13^0 - 4' = 12^0.56'$ very nearly; and by repeating the process we might obtain the value of a still more accurately.

Knowing a, we know $a = \dfrac{l}{\cos. a} = l \sec. a.$

To find the depth EC, to which the vertex hangs, we have, by Art. 49,

$$EC = BF - CD = a - a \sin. a = l.(\sec. a - \tan. a).$$

For c, the tension at the point C, we have, by the same Article,

$$c = a \sin. a = l \tan. a.$$

In the case just mentioned, where $l = 2h$, we shall have

$$a = 2.152h;$$

$$c = .459h;$$

$$EC = 1.693h.$$

If it were required to find the form of the curve when a is 45^0, it might be obtained directly from the formula; which gives in this case

$$\frac{l}{h} = 1.1346.$$

51. Prob. II. *A chain of given length APB, fig.* 118, *is suspended from two given points A, B, not in the same horizontal line; to find its position.*

Let s represent the whole length of the chain, and x and y the ordinates of the point B, measured from A; and therefore given quantities. By equation (3), Art. 49, we have

$$a^2 + 2as.\cos. \alpha + s^2 = (a+x)^2 = a^2 + 2ax + x^2;$$

$$\therefore a = \frac{s^2 - x^2}{2(x - s\cos. \alpha)}.$$

Hence $\sqrt{(a^2 + 2as.\cos.\alpha + s^2)} = a + x = \dfrac{s^2 - 2sx.\cos.\alpha + x^2}{2(x - s\cos.\alpha)}$;

$$a\cos.\alpha + s = \frac{2sx - s^2.\cos.\alpha - x^2\cos.\alpha}{2(x - s\cos.\alpha)}.$$

But by (6) Art. 49,

$$y = a.\sin.\alpha\, \mathrm{l}\, \frac{\sqrt{(a^2 + 2as.\cos.\alpha + s^2)} + (a.\cos.\alpha + s)}{a(1+\cos.\alpha)},$$

$$= \frac{(s^2 - x^2).\sin.\alpha}{2(x - s.\cos.\alpha)}\, \mathrm{l}\, \frac{(s^2 + 2sx + x^2).(1 - \cos.\alpha)}{(s^2 - x^2).(1 + \cos.\alpha)},$$

$$= \frac{(s^2 - x^2).\sin.\alpha}{2(x - s.\cos.\alpha)}\, \mathrm{l}\, \frac{s+x}{s-x}.\frac{1 - \cos.\alpha}{1 + \cos.\alpha};$$

whence α must be determined by approximation, as in the last problem. The approximation may be facilitated by the following artifice. Let $x = s.\cos.\beta$; hence

$$\frac{y}{s} = \frac{\sin.^2\beta.\sin.\alpha}{2(\cos.\beta - \cos.\alpha)}.\mathrm{l}\frac{1+\cos.\beta}{1-\cos.\beta}.\frac{1-\cos.\alpha}{1+\cos.\alpha}$$

$$=\frac{\sin.^2\beta\,.\,\sin.\,\alpha}{4\sin.\,\dfrac{\alpha+\beta}{2}\,.\,\sin.\,\dfrac{\alpha-\beta}{2}}\,\mathrm{l}\,\frac{\tan.\,\dfrac{\alpha}{2}}{\tan.\,\dfrac{\beta}{2}};$$

$$=\frac{\mathrm{Sin.}^2\beta\,.\,\mathrm{Sin.}\,\alpha}{4R\,.\,\mathrm{Sin.}\,\dfrac{\alpha=\beta}{2}\,.\,\mathrm{Sin.}\,\dfrac{\alpha-\beta}{2}}\,.\,M\,.\left\{\log.\,\mathrm{Tan.}\,\frac{\alpha}{2}-\log.\,\mathrm{Tan.}\,\frac{\beta}{2}\right\},$$

R being the radius, and Sin. &c. the sine, &c. of the tables.

Hence

$$\log.\,\frac{y}{s}=2\log.\,\mathrm{Sin.}\,\beta+\log.\,\mathrm{Sin.}\,\alpha-\log.\,\mathrm{Sin.}\,\frac{\alpha+\beta}{2}-\log.\,\mathrm{Sin.}\,\frac{\alpha-\beta}{2}$$

$$+\log.\left\{\log.\,\mathrm{Tan.}\,\frac{\alpha}{2}-\log.\,\mathrm{Tan.}\,\frac{\beta}{2}\right\}+\log.\,M-\log.\,4-10,$$

and by assuming values of α, and comparing the resulting with the true values of $\log.\,\frac{y}{s}$, we may obtain as before an answer nearly correct.

52. PROB. III. *A chain of given length $FBCB'F'=2l$, hangs freely over two given points B, B', in the same horizontal line, its ends BF, $B'F'$ hanging vertically: to find the position in which it will support itself.*

It is manifest that there cannot be an equilibrium except the two vertical parts BF, $B'F'$, are equal. Also each of these must be equal to the length which expresses the tension at B or B'; that is, $BF' = BF = \dfrac{s}{\cos.\,\alpha}$.

Let $BB' = 2h$, $CB = CB' = s$, the angle at $B = \alpha$, and we have, as before, Art. 50,

$$h = s\tan.\,\alpha\,\mathrm{l}\,\frac{1+\cos.\,\alpha}{\sin.\,\alpha}.$$

Also by Cor. 2, Art. 46, tension at $B = \dfrac{s}{\cos. a} = BF$;

$$\therefore l = CB + BF = s + \frac{s}{\cos. a} = s\frac{1 + \cos. a}{\cos. a};$$

$$\therefore \frac{h}{l} = \frac{\sin. a}{1 + \cos. a} \, \mathrm{l} \, \frac{1 + \cos. a}{\sin. a};$$

$$\text{or, } \frac{h}{l} = \tan. \frac{a}{2} . \, \mathrm{l} \, \text{co-tan.} \frac{a}{2} = - \tan. \frac{a}{2} \, \mathrm{l} \, \tan. \frac{a}{2};$$

whence $\tan. \frac{a}{2}$ must be found. And a being known, we know

$$s = l . \frac{\cos. a}{1 + \cos. a}: \text{ and } h = s \tan. a \, \mathrm{l} \frac{1 + \cos. a}{\sin. a};$$

and hence the curve is known.

53. PROB. IV. *In the last case, to find when the equilibrium is possible.*

In the equation $\frac{h}{l} = - \tan. \frac{a}{2} \, \mathrm{l} \, \tan. \frac{a}{2}$, making $\frac{l}{h} = u$, and $\tan. \frac{a}{2} = t$, we have $u = - \dfrac{1}{t \, \mathrm{l} \, t}$. And the relative changes of magnitude of t and u will be seen most easily by constructing a curve of which these shall be the abscissa and ordinate. Let bm, fig. 119, be always taken $= \tan. \frac{a}{2} = t$, and $mp = u$, and let us consider the locus of p.

The value of a will be between 0 and $\frac{1}{2}\pi$, and hence the value of t will be between 0 and 1; and hence $\mathrm{l}\, t$ will be negative, and u will always be positive

When $t = 0$, $t \, \mathrm{l} \, t = 0$, and u is infinite.

As t increases u decreases; we have

$$\frac{du}{dt} = \frac{1 + \mathrm{l} t}{(t \, \mathrm{l} \, t)^2};$$

which is negative so long as $- \mathrm{l} t > 1$.

When $1 + 1t = 0$, or $t = \frac{1}{e}$, $\frac{du}{dt} = 0$, and u is a minimum; at this point $u = \frac{l}{e}$, or $l = eh$. Afterwards u increases continually till $t = 1$, when u is infinite.

Hence the curve is of the form pqp', with asymptotes at b and c, bc being $= 1$. If $bn = .368$, &c. nq will be the minimum ordinate $= 2.718$, &c.

For every value, as bo, of u or $\frac{l}{h}$, there are two values of t, which may be found by drawing opp' parallel to bc. Hence there are *two* positions of equilibrium*, for given values of h and l. If we make bA perpendicular and equal to bc, and join Am, Am', the angles α for these two positions will be the doubles of bAm, bAm', respectively.

Thus, if $l = 10h$ the values of α are $3^0\, 12'$, and $83^0\, 36'$.

The least value of u or $\frac{l}{h}$ for which the equilibrium is possible, is when $u = e$, or $l = he$, which gives the minimum ordinate nq. In this case we have

$$t = \frac{1}{e}; \ \therefore \text{co-tan.} \frac{\alpha}{2} = \frac{1}{t} = 2.718281824, \text{ \&c.};$$

$$\therefore \frac{\alpha}{2} = 20^0\, 12', \text{ and } \alpha = 40^0\, 24'.$$

If $a = BF$, fig. 118,

$$a = \frac{s}{\cos.\alpha} = \frac{l}{1 + \cos.\alpha} = \frac{l}{2\cos.^2 \frac{\alpha}{2}} = \frac{l}{2}\left(1 + \frac{1}{e^2}\right);$$

---

* In the case when the equilibrium is possible, the higher position is one of *stable*, the lower one of *unstable* equilibrium. If the chain be placed with its vertex above the higher of the two positions of equilibrium, it will descend towards that: if it be placed any where between the two positions, it will ascend towards the upper. If it be placed below the lower it will descend and never come to another position of equilibrium.

$$s = l - 2a = \frac{l}{2}\left(1 - \frac{1}{e^2}\right);$$

and by Art. 49, if k be the depth of the vertex below the horizontal line,

$$k = a - a \,.\, \sin.\, \alpha = \frac{l}{2} \,.\, \frac{1 - \sin.\, \alpha}{\cos.^2 \frac{\alpha}{2}} = \frac{l}{2}\left(\sec.^2 \frac{\alpha}{2} - 2 \tan.\, \frac{\alpha}{2}\right);$$

$$= \frac{l}{2} \,.\, \left(1 + \frac{1}{e^2} - \frac{2}{e}\right) = \frac{l}{2}\left(1 - \frac{1}{e}\right)^2,$$

$$\frac{s}{k} = \frac{e + 1}{e - 1}.$$

If the chain be so short, compared with the distance, that $\frac{l}{h}$ is less than e, it cannot be supported: the middle part will descend and draw up the ends.

54. Prob. V. *To find the center of gravity of the catenary AP*, fig. 118.

For this purpose we must find

$$\int_x x \frac{ds}{dx}, \;\; \int_y y \frac{ds}{dy}.$$

By Cor. Art. 49,

$$\frac{dy}{dx} = \frac{c}{s'} = \frac{a \sin.\, \alpha}{a \cos.\, \alpha + s};$$

$$\therefore\; a \cos.\, \alpha + s = a \sin.\, \alpha \,.\, \frac{dx}{dy};$$

$$\therefore\; s = a\left(\sin.\, \alpha \,.\, \frac{dx}{dy} - \cos.\, \alpha\right),$$

$$\int_y s = a\,(x \sin.\, \alpha - y \cos.\, \alpha);$$

$$\therefore\; \int_y y \frac{ds}{dy} = ys - \int_y s = ys - a\,(x \sin.\, \alpha - y \cos.\, \alpha).$$

Again, since by (2) and (5), Art. 49,

$$\frac{dx}{ds'} = \frac{s'}{\sqrt{(c^2 + s'^2)}}, \quad \frac{dy}{ds'} = \frac{c}{\sqrt{(c^2 + s'^2)}};$$

$$\sqrt{(c^2 + s'^2)} = s' \frac{dx}{ds'} + c \frac{dy}{ds'};$$

and substituting $\dfrac{dx}{ds}$, $\dfrac{dy}{ds}$ for $\dfrac{dx}{ds'}$, $\dfrac{dy}{ds'}$;

or, putting for c and s' their values,

$$\sqrt{(a^2 + 2as \cos. \alpha + s^2)} = (a \cos. \alpha + s) \frac{dx}{ds} + a \sin. \alpha . \frac{dy}{ds}.$$

Hence by (3), Art. 49,

$$\int_s x \frac{ds}{dx} = \int \{\sqrt{(a^2 + 2as \cos. \alpha + s^2)} - a\} \frac{ds}{dx}$$

$$= \int_s \left\{(a \cos. \alpha + s) \frac{dx}{ds} + a \sin. \alpha . \frac{dy}{ds} - a \frac{ds}{dx}\right\}.$$

Add $\int_s x \dfrac{ds}{dx}$ to both sides, and integrate, observing that

$$\int_s \left(s + x \frac{ds}{dx}\right) = xs;$$

$$\therefore 2\int_s x \frac{ds}{dx} = ax \cos. \alpha + xs + ay \sin. \alpha - as.$$

Hence by the formulæ, Art. 34,

$$h = \frac{\int_s x \dfrac{ds}{dx}}{s} = \frac{a(x \cos. \alpha + y \sin. \alpha)}{2s} - \frac{a - x}{2},$$

$$k = \frac{\int_y y \dfrac{ds}{dy}}{s} = y - \frac{a(x \sin. \alpha - y \cos. \alpha)}{s}.$$

Cor. 1. It may be observed, that if we draw PO perpendicular on the tangent at P,

$$AO = x \cos. a + y \sin. a,$$

$$PO = x \sin. a - y \cos. a.$$

Cor. 2. If we suppose A to be the lowest point, we have a a right angle. Hence

$$a + h = \frac{ay}{2s} + \frac{a + x}{2}.$$

which agrees with Art. 34, Ex. 23.

Cor. 3. Let the tangents at A and P meet in T, and let TU be vertical; $PU = u$; $\therefore NU = y - u$, and if $PTU = \theta$, we shall have

$$u \text{ co-tan. } \theta + (y - u) \text{ co-tan. } a = x.$$

$$\text{But co-tan. } \theta = \frac{dx}{dy} = \frac{a \cos. a + s}{a \sin. a}$$

$$= \text{co-tan. } a + \frac{s}{a \sin. a};$$

$$\therefore \frac{us}{a \sin. a} + \frac{y \cos. a}{\sin. a} = x,$$

$$u = \frac{a\,(x \sin. a - y \cos. a)}{s};$$

$\therefore k + u = y$, and the center of gravity is in the vertical line TU which passes through T.

2. *The Catenary when the force acts in parallel lines and the Chain is not uniform.*

55. We may consider the thickness of the chain or cord to be variable, supposing it still to be so small throughout that we may consider the flexible body as a physical line. Or we may conceive the catenary to be a surface of unequal

breadth, resembling a ribbon, its breadth being parallel to the horizon; so that it may be a portion of a cylindrical surface, the curve of the cylinder being the catenary. We may also suppose the density to be variable. Or we may conceive the force which acts upon the chain, and gives weight to it, to be different in different parts.

Upon any of these suppositions the weight of equal portions of the curve taken in different parts of it will be different. Let $\frac{ds}{dy}$ be the differential coefficient of the curve with respect to y, and let $w\frac{ds}{dy}$ be the differential coefficient of the weight; w being the quantity (thickness, breadth, density or force) to which the weight of a given element of length is proportional.

Hence $\int_y w\frac{ds}{dy}$ taken between proper limits is the weight of any portion of the chain or string.

PROP. *To find the curve when the law of the thickness is given, and conversely.*

In fig. 117, let C be the lowest point, and let ma be the tension there. Then x and y being CM and MP as before, we shall have, as in Art. 46,

$$\frac{dx}{dy} = \frac{\int_y w\frac{ds}{dy}}{ma} \ldots\ldots\ldots\ldots(1).$$

If we differentiate this with respect to y, we have

$$\frac{d^2x}{dy^2} = \frac{w}{ma}\cdot\frac{ds}{dy} \ldots\ldots\ldots\ldots(2).$$

And w being known in terms of the other variable quantities, we shall, by integrating, have the equation to the curve.

$$\text{Also } w = ma\cdot\frac{d^2x}{dy^2}\cdot\frac{dy}{ds} \ldots\ldots\ldots(3).$$

whence, if the curve be known, we may, by differentiating, find w.

56. PROB. VI. *A flexible string, whose thickness at every point is inversely as the square root of the length measured from the lowest point, is acted upon by gravity; to find its form.*

Let m be the thickness at a length c from the lowest point; hence, at the end of a length s,

$$w = m\frac{\sqrt{c}}{\sqrt{s}}; \quad \int_y w\frac{ds}{dy} = m\int_y \frac{\sqrt{c}}{\sqrt{s}}\cdot\frac{ds}{dy} = 2m\sqrt{(cs)};$$

$$\therefore \text{ by (1) } \frac{dx}{dy} = \frac{2\sqrt{(cs)}}{a}; \quad \left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = \frac{a^2+4cs}{a^2};$$

$$\frac{1}{a}\cdot\frac{dy}{ds} = \frac{1}{\sqrt{(a^2+4cs)}}; \text{ whence } \frac{y}{a} = \frac{\sqrt{(a^2+4cs)}}{2c} - \frac{a}{2c};$$

$$\left(\frac{y}{a}+\frac{a}{2c}\right)^2 = \frac{a^2}{4c^2} + \frac{s}{c}; \quad \frac{s}{c} = \frac{y^2}{a^2} + \frac{y}{c}.$$

$$\text{Hence } \frac{1}{c}\cdot\frac{ds}{dy} = \left(\frac{2y}{a^2}+\frac{1}{c}\right); \quad \frac{1}{c^2}\left\{1+\left(\frac{dx}{dy}\right)^2\right\} = \left(\frac{2y}{a^2}+\frac{1}{c}\right)^2;$$

$$\frac{1}{c^2}\left(\frac{dx}{dy}\right)^2 = \left(\frac{4y^2}{a^4}+\frac{4y}{a^2c}\right); \quad \frac{dx}{dy} = \frac{2c}{a^2}\sqrt{\left(y^2+\frac{1}{c}y\right)};$$

whence y is easily found by integrating; and hence the curve is known.

57. PROB. VII. *A flexible string is acted on by a force which is, at every point, as the height above the lowest point: to find its form.*

Let the origin be as before: and at the height c above the lowest point let the force be m; hence, at the height x, since *cæteris paribus* the weight of any portion will be as the force,

$$w = \frac{mx}{c}; \quad \therefore \text{ by (1) } \frac{dx}{dy} = \frac{\int_y x\frac{ds}{dy}}{ca}, \quad \frac{d^2x}{dy^2} = \frac{x}{ca}\cdot\frac{ds}{dy};$$

$$\frac{\frac{d^2x}{dy^2}}{\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}}=\frac{x}{ca};\quad \frac{\frac{dx}{dy}\cdot\frac{d^2x}{dy^2}}{\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}}=\frac{x}{ca}\cdot\frac{dx}{dy};$$

Integrating with respect to y,

$$\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}=\frac{x^2}{2ca}+1;\quad \frac{dx}{dy}=\sqrt{\left(\frac{x^4}{4c^2a^2}+\frac{x^2}{ca}\right)};$$

$$\frac{dy}{dx}=\frac{2ac}{x\sqrt{(x^2+4ac)}};\quad y=\frac{\sqrt{(ac)}}{2}\,\mathrm{l}\,\frac{\sqrt{(x^2+4ac)}-2\sqrt{(ac)}}{\sqrt{(x^2+4ac)}+2\sqrt{(ac)}}+\text{const.}$$

When $x=0$, y is infinite and negative; when x is infinite, y is equal to the constant. Hence the curve has a vertical and a horizontal asymptote, and never meets the horizontal line in which the force is $=0$.

58. Prob. VIII. *To find the law of thickness of a string that it may hang in the form of a semi-circle.*

Placing the origin at the lowest point, as before, we must have, calling the radius of the circle c,

$$y=\sqrt{(2cx-x^2)};\quad \therefore\ \frac{dx}{dy}=\frac{\sqrt{(2cx-x^2)}}{c-x},\quad \frac{ds}{dy}=\frac{c}{c-x};$$

$$\text{also, }\frac{d^2x}{dy^2}=\frac{d\left(\frac{dx}{dy}\right)}{dy}=\frac{d\left(\frac{dx}{dy}\right)}{dx}\cdot\frac{dx}{dy}=\frac{c^2}{(c-x)^2\sqrt{(2cx-x^2)}}\,\frac{dx}{dy}$$

$$=\frac{c^2}{(c-x)^3};\quad\text{also }\frac{dy}{ds}=\frac{c-x}{c};$$

$$\text{whence by (3), }w=ma\cdot\frac{d^2x}{dy^2}\cdot\frac{dy}{ds}=\frac{mac}{(c-x)^2};$$

hence the thickness must vary inversely as the square of the depth below the horizontal diameter.

The tension will be found, as before, by the equation,

$$\text{tension}=ma\cdot\frac{ds}{dy}=\frac{mac}{c-x}.$$

Hence, at the extremities of the horizontal diameter it is infinite.

If, instead of supposing the thickness of the string to vary, we suppose to be hung to each point of it vertical strings of uniform thickness whose lengths are proportional to

$$\frac{mac}{(c-x)^2},$$

the curve which it will form will be the same. And this is also applicable to all the cases of this section.

59. PROB. IX. *To find the law of thickness of a string that it may hang in the form of a parabola with its axis vertical.*

The origin is at the lowest point as before:

$$\text{By (3)}, \quad w = ma \cdot \frac{d^2x}{dy^2} \cdot \frac{ds}{dy},$$

$$\text{and } y^2 = 4cx; \ \therefore \ \frac{dy}{dx} = \frac{\sqrt{c}}{\sqrt{x}}; \ \frac{ds}{dy} = \frac{\sqrt{(x+c)}}{\sqrt{c}};$$

$$\frac{d^2x}{dy^2} = \frac{1}{2\sqrt{(cx)}} \cdot \frac{dx}{dy} = \frac{1}{2c};$$

$$\therefore \ w = \frac{ma}{2\sqrt{(cx+c^2)}}.$$

When $x = 0$, $w = \dfrac{ma}{2c}$; and if m be the thickness at the lowest point, $a = 2c$,

$$w = \frac{m\sqrt{c}}{\sqrt{(x+c)}}.$$

So long as x is small, this is nearly constant. Hence, conversely, if the thickness be constant, the catenary, within a small distance of the vertex, nearly coincides with a parabola. This is a conclusion to which Galileo was led by experiment.

3. *The Catenary when the Chain is acted upon by a central attractive or repulsive force**.

60. PROP. *To find the equation to the catenary when the force tends to a center.*

Let S, fig. 120, be the center of attractive force, and at any distance $SP = r$, let the force be $= f$, f being a function of r. Let $AP = s$ be the chain or cord, and at the point P let the mass of a small particle δs be $\mu \delta s$, μ depending upon the thickness, density, &c.

Let A be the point at which the curve is perpendicular to SA. Make SA a line of abscissas, and let MP be an ordinate perpendicular to it: Py a tangent at P, and Sy perpendicular on it.

Put $Sy = p$; tension at $A = a$, tension at $P = t$; angle $ASP = \theta$, $ATP = \phi$. The weight of a particle ds at P will be $f\mu$ in the direction PS, (see Note:) and if we resolve this force in the directions parallel and perpendicular to AS, the components will be $f\mu \cos. \theta$ and $f\mu \sin. \theta$: and hence the whole effects of the weight in those directions will be $\int_s f\mu \cos. \theta$ and $\int_s f\mu \sin. \theta$. The other forces which act on the cord AP, are the tension at $A = a$; and the tension at $P = t$, which may be resolved into the parts $t . \cos. \phi$ and $t . \sin \phi$, in the rectangular directions. As before, the forces which keep AP at rest must be subject to the conditions of Art. 22. Hence,

$$\left.\begin{aligned} \int_s f\mu \cos. \theta &= t \cos. \phi \\ \int_s f\mu \sin. \theta &= t \sin. \phi - a \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating with respect to s,

$$f\mu \cos. \theta = \frac{dt}{ds} . \cos. \phi - t \sin. \phi \frac{d\phi}{ds},$$

* The force spoken of here and in the last Article is the attractive force which produces weight or pressure in the bodies on which it acts. If other things remain the same, such attractive forces are as the weight which they produce in a given particle of matter.

$$f\mu \sin.\theta = \frac{dt}{ds} \cdot \sin.\phi + t\cos.\phi\frac{d\phi}{ds}.$$

Multiply the first by cos. ϕ, and the second by sin. ϕ, and add; and we have

$$f\mu\cos.(\phi-\theta) = \frac{dt}{ds} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Multiplying the first by sin. ϕ, and the second by cos. ϕ, and subtract; and we have

$$f\mu\sin.(\phi-\theta) = -t\frac{d\phi}{ds}\ldots\ldots\ldots\ldots\ldots (3).$$

But $\phi - \theta = ATP - TSP = SPy$. Also if we take PQ a small arc, and draw Qn perpendicular on SP, $\frac{Pn}{PQ}$, $\frac{Qn}{PQ}$, are ultimately as $\frac{dr}{ds}$, $r\frac{d\theta}{ds}$. Hence

$$\cos.(\phi-\theta) = \frac{dr}{ds},\quad \sin.(\phi-\theta) = r\frac{d\theta}{ds},$$

and (2) and (3) become

$$f\mu\frac{dr}{ds} = \frac{dt}{ds},\ \text{ or } f\mu = \frac{dt}{dr}\ldots\ldots\ldots\ldots\ldots (4)$$

$$f\mu r\frac{d\theta}{ds} = -t\frac{d\phi}{ds} \ldots \text{ or } f\mu r = -t\frac{d\phi}{d\theta}\ldots (5).$$

Now $r\dfrac{d\theta}{dr} = \dfrac{Qn}{Pn} = \dfrac{p}{\sqrt{(r^2-p^2)}}$;

$$\therefore \frac{d\theta}{dr} = \frac{p}{r\sqrt{(r^2-p^2)}}.$$

Also $\phi - \theta = \text{arc.}\left(\sin. = \dfrac{p}{r}\right)$;

$$\therefore \frac{d\phi}{dr} - \frac{d\theta}{dr} = \frac{r\dfrac{dp}{dr} - p}{r\sqrt{(r^2-p^2)}};$$

$$\therefore \text{ adding, } \frac{d\phi}{dr} = \frac{\frac{dp}{dr}}{\sqrt{(r^2 - p^2)}}.$$

Also (5) is equivalent to

$$f\mu r \frac{d\theta}{dr} = -t \frac{d\theta}{dr}.$$

Putting the values of $\frac{d\theta}{dr}$ and $\frac{d\phi}{dr}$ in this, it becomes, reducing,

$$f\mu p = -t \frac{dp}{dr} \dots\dots\dots\dots\dots\dots (6).$$

Dividing (6) by (4) we have

$$p = -\frac{t \frac{dp}{dr}}{\frac{dt}{dr}};$$

$$\therefore p \frac{dt}{dr} + t \frac{dp}{dr} = 0;$$

$$pt = C^* \dots\dots\dots\dots\dots\dots\dots (7),$$

C being a constant quantity, to be determined by the conditions of the question.

Also we have, by (4),

$$t = \int_r f\mu.$$

$$\text{Hence } p = \frac{C}{\int_r f\mu}.$$

When f is known in terms of r, this equation gives the curve AP, by an equation between the distance SP and the

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* The property, that the perpendicular is inversely as the tension, appears also from this, that AP is acted on by the tensions at A and P, and also by central forces all tending to S. Hence the result of these latter forces will also tend to S; and hence we may suppose AP retained by a lever passing through S as a fulcrum, and the two forces at A and at P will be inversely as the perpendiculars on their directions; therefore tension at $P . Sy$ = tension at $A . SA$ = a constant quantity.

perpendicular Sy upon the tangent. And from this equation we may determine the relation between r and θ; and between x and y: unless this is rendered impossible by the difficulty of integrating.

The tension $t = \int_r f\mu$; if the thickness and density be constant, we may make $\mu = 1$, and $t = \int_r f$; hence the tension depends only on the distance r, and is not affected by the form of the curve. If we suppose the end Pp to hang freely over the point P, and thus to produce the equilibrium, its weight must be $\int_r f$; which is also the weight of a string extending from a point p, at a given distance from S, up to P. Hence at every point P the string Pp will hang to the same distance Sp from S; and the ends of all the strings will be in a circle with center S; in which circle also is the point a, Aa being the length whose weight is requisite to produce the tension at A.

61. Prob. X. *The force varying inversely as the square of the distance from S, it is required to find the form of the catenary.*

Let SA, fig. 120, $= c$, and the force at $A = k$; hence

$$f = \frac{kc^2}{r^2};$$

$$t = \int_r f = \int_r \frac{kc^2}{r^2} = \text{constant} - \frac{kc^2}{r} = a + kc - \frac{kc^2}{r};$$

for when $r = c$, $t = a$.

Hence $$p = \frac{C}{a + kc - \dfrac{kc^2}{r}} = \frac{ac}{a + kc - \dfrac{kc^2}{r}},$$

for when $r = c$, $p = c$,

$$= \frac{acr}{(a + kc)\,r - kc^2}.$$

Let $a = nkc$; kc being the weight of a length of string AS, acted on by a constant force equal to that at A; hence

$$p = \frac{ncr}{(n + 1)\,r - c}.$$

To determine the nature of the curve, we have

$$\frac{d\theta}{dr}=\frac{1}{r\sqrt{\left(\frac{r^2}{p^2}-1\right)}}=\frac{nc}{r\sqrt{\{(n+1)^2r^2-2(n+1)cr+c^2-n^2c^2\}}};$$

which will give different forms as n is greater than, equal to, or less than unity.

(1.) Let $n>1$; therefore

$$\frac{d\theta}{dr}=\frac{nc}{r^2\sqrt{\left((n+1)^2-2(n+1)\cdot\frac{c}{r}-(n^2-1)\frac{c^2}{r^2}\right)}};$$

which may be integrated by making $\frac{n-1}{n}\cdot\frac{c}{r}+\frac{1}{n}=u$, and gives

$$\theta=\frac{n}{\sqrt{(n^2-1)}}\cdot\text{arc.}\left(\cos.=\frac{(n-1)c+r}{nr}\right);$$

θ being measured from the line SA.

If we make r infinite, we have

$$\theta=\frac{n}{\sqrt{(n^2-1)}}\cdot\text{arc.}\left(\cos.=\frac{1}{n}\right);$$

which gives the position of the asymptotes of the curve.

The angle which the asymptotes make with SA, is greater, as n, and consequently the tension at A, is greater. When n is infinite, it is a right angle, and the curve becomes a straight line perpendicular to SA. As n diminishes to unity, θ diminishes to the value which it has in the next case.

(2.) Let $n=1$: hence

$$\frac{d\theta}{dr}=\frac{c}{2r\sqrt{(r^2-cr)}}=\frac{c}{2r^2\sqrt{\left(1-\frac{c}{r}\right)}};$$

which gives, by integrating,

$$\theta = \sqrt{\left(1 - \frac{c}{r}\right)} + \text{const.} = \sqrt{\left(1 - \frac{c}{r}\right)};$$

because $\theta = 0$ when $r = c$.

When r is infinite $\theta = 1$. Hence the angle which the asymptotes make with SA is that whose arc is equal to the radius; or, if RO be the asymptote,

$$ARO = 57^0\ 14'\ 44''\ 48'''.$$

In every case we may find the position of the asymptote by making r infinite in the value of p; which will give

$$Sz = \frac{nc}{n+1}.$$

(3.) Let $n < 1$: hence

$$\frac{d\theta}{dr} = \frac{nc}{r^2\sqrt{\left((1+n)^2 - 2(1+n)\frac{c}{r} + (1-n^2)\frac{c^2}{r^2}\right)}};$$

which may be integrated by making $1 - (1-n)\dfrac{c}{r} = u$; and gives $\theta =$

$$\frac{n}{\sqrt{(1-n^2)}}\,\mathrm{l}\,\frac{r - (1-n)c + \sqrt{\{(1-n^2)r^2 - 2cr + (1-n)^2c^2\}}}{nr};$$

the integral being corrected so as to vanish when $r = c$.

When r is infinite, $\theta = \dfrac{n}{\sqrt{(1-n^2)}}\,.\,\mathrm{l}\,\dfrac{1+\sqrt{(1-n^2)}}{n}$, which gives the position of the asymptote. When $n = 1$, $\theta = 1$, as may easily be shewn, agreeably to the last case. As n diminishes, the angle which the asymptotes make with SA diminishes, and when n becomes 0 this angle vanishes.

The tension at A is equal to the weight of a string whose length is nc, acted upon by a constant force equal to that at A. But if $Sa = b$, the weight of the portion Aa, acted

on by the variable force (which weight expresses the tension at A) will be

$$= \int_r \frac{kc^2}{r^2}, \text{ the integral taken from } r = b, \text{ to } r = c$$

$$= \frac{kc^2}{b} - \frac{kc^2}{c} = nkc, \text{ by supposition;}$$

$$\therefore b = \frac{c}{1+n}.$$

Hence if a circle were described with a radius $Aa = b$, the string hanging down from any point of the curve, must, in order to produce the tension at that point, reach to the circumference of this circle.

62. Prob. XI. *Let the force vary as the* m[th] *power of the distance from* S: *to find the curve.*

Retaining the notation of the last Problem, we have

$$\text{force at } P = \frac{kr^m}{c^m}; \ \therefore t = \int_r \frac{kr^m}{c^m} = \frac{kr^{m+1}}{(m+1)c^m} + \text{const.}$$

$$= a - \frac{kc}{m+1} + \frac{kr^{m+1}}{(m+1)c^m};$$

a being the tension at A. Let $a = \dfrac{nkc}{m+1}$; therefore

$$t = \frac{kc}{m+1}\left(n - 1 + \frac{r^{m+1}}{c^{m+1}}\right).$$

Hence

$$p = \frac{nc}{n - 1 + \dfrac{r^{m+1}}{c^{m+1}}} = \frac{nc^{m+2}}{(n-1)c^{m+1} + r^{m+1}}:$$

and

$$\frac{d\theta}{dr} = \frac{1}{r\sqrt{\left(\frac{r^2}{p^2} - 1\right)}} = \frac{nc^{m+2}}{r\sqrt{\left[\left\{r^{m+2} + (n-1)c^{m+1}r\right\}^2 - n^2c^{2m+4}\right]}},$$

which cannot be integrated generally except $n = 1$.

In the case of $n = 1$,

$$\frac{d\theta}{dr} = \frac{c^{m+2}}{r\sqrt{\left\{r^{2m+4} - c^{2m+4}\right\}}};$$

which may be integrated by making $c^{m+2}u = r^{m+2}$: this substitution gives

$$\frac{d\theta}{dr} = (m + 2)\, u\sqrt{(u^2 - 1)};$$

$$\therefore\ \theta = \frac{1}{m + 2}.\text{arc}\,(\text{sec.} = u)$$

$$= \frac{1}{(m + 2}\text{arc}\left(\text{sec.} = \frac{r^{m+2}}{c^{m+2}}\right);$$

$$\frac{r^{m+2}}{c^{m+2}} = \text{sec.}\,(m + 2)\theta;\quad r^{m+2}\cos.\,(m + 2)\,\theta = c^{m+2}.$$

If we make r infinite, we have for the inclination of the asymptotes to SA,

$$\theta = \frac{\pi}{2m + 4}.$$

63. We may find the equation between

$$SM = x \text{ and } MP = y.$$

For
$$r = \sqrt{(x^2 + y^2)},$$

$$\cos.\theta = \frac{x}{\sqrt{(x^2 + y^2)}},\quad \tan.\theta = \frac{y}{x}.$$

Hence

$$c^{m+2} = r^{m+2} \cos. (m+2)\,\theta$$

$$= r^{m+2}. \left(\cos.^{m+2}\theta - \frac{(m+2)(m+1)}{1.2}\cos.^{m}\theta . \sin^{2}.\theta\right.$$

$$\left. + \frac{(m+2)(m+1)m(m-1)}{1.2.3.4}\cos.^{m-2}\theta . \sin.^{4}\theta ......\right)$$

$$= r^{m+2}\cos.^{m+2}\theta . \left(1 - \frac{(m+2)(m+1)}{1.2}\tan.^{2}\theta\right.$$

$$\left. + \frac{(m+2)......(m-1)}{1.2.3.4}\tan.^{4}\theta - ......\right)$$

$$= x^{m+2}\left(1 - \frac{(m+2)(m+1)}{1.2} . \frac{y^2}{x^2} + \frac{(m+2)...(m-1)}{1.2.3.4} . \frac{y^4}{x^4} - ...\right).$$

Cor. 1. If $m = 0$, or the force be constant,

$$c^2 = x^2\left(1 - \frac{y^2}{x^2}\right) = x^2 - y^2.$$

Hence the curve is the rectangular hyperbola. The asymptotes make angles of 45^0 with SA.

Cor. 2. If $m = 1$, or the force be as the distance,

$$c^3 = x^3\left(1 - \frac{3y^3}{x^3}\right) = x^3 - 3y^3.$$

In this case the angle which the asymptotes make with SA is $34^0\,44'$.

64. Prop. *To find the catenary when the central force is repulsive.*

The process for finding the curve of equilibrium in this case will be nearly the same as before, with the excep-

tion of the signs of some of the quantities, and the results will be

$$-f\mu = \frac{dt}{dr};$$

$$f\mu p = t\frac{dp}{dr}:$$

$$\therefore \text{ dividing, } \quad \frac{\frac{dp}{dr}}{p} = \frac{\frac{dt}{dr}}{t}, \quad \therefore p = \frac{C}{t},$$

$$\text{and } t = \textstyle\int_r f\mu; \quad \therefore p = \frac{C}{-\int_r f\mu}.$$

65. Prob. XII. *S, fig.* 121, *is a center of repulsive force varying inversely as the square of the distance from S: to find the form of the curve AP, formed by a flexible string.*

Retaining the notation of Prob. X,

$$t = -\int_r \frac{kc^2}{r^2} = a - kc + \frac{kc^2}{r};$$

a being the tension at A: let $a = nkc$;

$$\therefore t = kc\left((n-1) + \frac{c}{r}\right).$$

Hence $p = \dfrac{ncr}{(n-1)\,r + c}$, supposing the curve at A perpendicular to SA;

$$\therefore \frac{d\theta}{dr} = -\frac{1}{r\sqrt{\left(\frac{r^2}{p^2} - 1\right)}}$$

$$= -\frac{nc}{r\sqrt{\{(n-1)^2 r^2 + 2(n-1)\,cr + c^2 - n^2c^2\}}},$$

which may be integrated nearly as in Prob. X.

If we suppose the curve not to be perpendicular to SA, but to make with it an angle a, we shall have at that point $p = c \,.\, \sin. a$;

$$\therefore p = \frac{ncr \,.\, \sin. a}{(n-1)\,r + c}.$$

If $n = 1$, this becomes $p = r \,.\, \sin. a$, and the curve is the *logarithmic spiral.*

66. Prob. XIII. *Let the force be inversely as the* m[th] *power of the distance: to find the curve.*

$$t = -\int_r \frac{kc^m}{r^m} = a - \frac{kc}{m-1} + \frac{kc^m}{(m-1)\,r^{m-1}};$$

$$= \frac{kc}{m-1} \,.\, \left(n - + \frac{c^{m-1}}{r^{m-1}}\right);$$

$$\text{putting } a = \frac{nkc}{m-1}.$$

Take the case when $n = 1$, and we have

$$t = \frac{kc^{m-1}}{(m-1)\,r^{m-1}}; \text{ and let } p = c \,.\, \sin. a, \text{ at } A;$$

$$\therefore p = \frac{C}{t} = \frac{r^{m-1} \,.\, \sin. a}{c^{m-2}};$$

$$\frac{d\theta}{dr} = -\frac{1}{r\sqrt{\left(\frac{r^2}{p^2} - 1\right)}} = -\frac{r^{m-2} \sin. a}{r\sqrt{(c^{2m-4} - r^{2m-4} \sin.^2 a)}}$$

$$= -\frac{r^{m-3} \sin. a}{\sqrt{(c^{2m-4} - r^{2m-4} \sin.^2 a)}}.$$

To integrate, put $\dfrac{r^{m-2} \sin. a}{c^{m-2}} = u$;

$$\therefore \frac{d\theta}{du} = -\frac{1}{(m-2)\sqrt{(1-u^2)}},$$

$$\theta = \frac{1}{m-2}\,\text{arc}\,(\cos. = u) = \frac{1}{m-1}\,\text{arc}\left(\cos. = \frac{r^{m-2}\sin.\alpha}{c^{m-2}}\right).$$

$$\text{Or, if } \frac{c^{m-2}}{\sin.\alpha} = a^{m-2},\ (m-2)\,\theta\,\text{arc}\left(\cos. = \frac{r^{m-2}}{a^{m-2}}\right).$$

Here a is the value of r at the point A.

To find the angle to $2ASO$ which comprehends the whole curve, make $r = 0$:

$$\therefore\ \theta = \frac{\pi}{2m-4};\quad \therefore\ 2ASO = 2\theta = \frac{\pi}{m-2}.$$

We may find the equation between $SM = x$, and $MA = y$, as before,

For $r^{m-2} = a^{m-2}\cos.(m-2)\,\theta$

$$= a^{m-2}\left(\cos.^{m-2}\theta - \frac{(m-2)\,(m-3)}{1\,.\,2}\cos.^{m-4}\theta\,.\,\sin.^2\theta + \ldots\right);$$

$$\therefore\ r^{2m-4} = a^{m-}r^{2m-2}\cos.^{m-2}\theta\left(1 - \frac{(m-2)\,(m-3)}{1\,.\,2}\tan.^2\theta + \ldots\right);$$

$$\text{or } (x^2+y^2)^{m-2} = a^{m-2}x^{m-2}\left(1 - \frac{(m-2)\,(m-3)}{1\,.\,2}\,.\,\frac{y^2}{x^2} + \ldots\right).$$

Cor. 1. If $m = 3$, $\theta = \text{arc}\left(\cos. = \dfrac{r}{a}\right)$; hence APS is a circle on the diameter AS.

Cor. 2. If $m = 4$, $2\theta = \text{arc}\left(\cos. = \dfrac{r^2}{a^2}\right)$; hence APS is the lemniscata with its knot at S.

Cor. 3. Hence if there be a centre of repulsive force which varies inversely as the cube of the distance, and if the two ends of a string be fastened at this center, it will form itself into a circle. If the force vary inversely as the fourth power, the curve will be a lemniscata, and so on.

4. *The Catenary when the Chain is acted upon by any Forces.*

67. PROP. *Let forces to act upon the flexible body AP, fig.* 122, *in the same plane, accordingto any law whatever; it is required to find its form.*

Let the force at any point P be represented by f, and act in the direction PF, which makes with the line of abscissas AM an angle ψ. The reasoning is exactly the same as in Art. 60. The effect of the force f at P is fds, and this, resolved parallel and perpendicular to AM, gives $f\cos.\psi$, and $f\sin.\psi$. Hence the whole effects on AP are $\int_s f\cos.\psi$, and $\int_s f\sin.\psi$. The remaining forces are the tension at P, which is represented by t, and makes with AM an angle ϕ, and the tension at A, which is represented by a, and is supposed to be perpendicular to AM. Hence the conditions of Art. 23, give

$$\int_s f\cos.\psi - t\cos.\phi = 0,$$

$$\int_s f\sin.\psi + t\sin.\phi = a.$$

Differentiating,

$$f\cos.\psi - \frac{dt}{ds}\cos.\phi + t\sin.\phi.\frac{d\phi}{ds} = 0.$$

$$f\sin.\psi + \frac{dt}{ds}\sin.\phi + t\cos.\phi.\frac{d\phi}{ds} = 0.$$

Multiply the first by $\cos.\phi$ and the second by $\sin.\phi$, and subtract: also multiply the first by $\sin.\phi$ and the second by $\cos.\phi$, and add: we shall thus get

$$f(\cos.\phi.\cos.\psi - \sin.\phi.\sin.\psi) - \frac{dt}{ds} = 0;$$

$$f(\sin.\phi.\cos.\psi + \cos.\phi.\sin.\psi) + t\frac{d\phi}{ds} = 0;$$

$$\text{or, } f\cos.(\phi+\psi) = \frac{dt}{ds},$$

$$f\sin.(\phi+\psi) = -t\frac{d\phi}{ds}.$$

The angle $\phi + \psi$ is FPT, the angle which the force makes with the tangent. This angle and the force f being expressed in terms of x and y and their differentials, t is known from the first equation: and this value of t, and $\frac{d\phi}{ds}$, being substituted in the second, we have the equation to the curve. For $\frac{d\phi}{ds}$, we have

$$\phi = \text{arc}\left(\tan. = \frac{dy}{dx}\right);$$

$$\therefore \frac{d\phi}{ds} = \frac{d\phi}{dx}\frac{dx}{ds} = \frac{\frac{d^2y}{dx} \cdot \frac{dx}{ds}}{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\frac{d^2y}{dx^2}}{\left(\frac{ds}{dx}\right)^3};$$

the differentiations being performed with respect to x.

68. Prop. *If the force be at every point perpendicular to the curve, to find the form.*

We shall have $\phi + \psi = \frac{1}{2}\pi$; hence $\cos.(\phi + \psi) = 0$, $\sin.(\phi + \psi) = 1$; and our equations become

$$0 = \frac{dt}{ds};$$

$$f = -t\frac{d\phi}{ds},$$

$$\text{or } t = a;$$

$$f = -a.\frac{\frac{d^2y}{dx^2}}{\left(\frac{ds}{dx}\right)^3} = -\frac{a}{\rho}; \quad \therefore a = f\rho,$$

ρ being the radius of curvature.

Hence *when the force is perpendicular to the curve, the tension is constant;* and is at every point equal to the weight of a portion of the cord, whose length is the radius of curvature, acted on by the force at that point. If curvature be

supposed inversely proportional to the radius of curvature, *the curvature at every point will be as the force.*

69. Prob. XIV. *A flexible line AP, fig.* 122, *is acted upon at every point P by a force* f, *which is perpendicular to the line, and which is as the square of the sine of the angle EPV; to find the curve AP.*

The sine of $EPV = \sin.\phi = \frac{dy}{ds}$; hence the force

$$f = k\left(\frac{dy}{ds^2}\right)^2,$$

k being its value at A;

$$\therefore\ k.\left(\frac{dy}{ds^2}\right)^2 = -a\frac{\frac{d^2y}{dx^2}}{\left(\frac{ds}{dx}\right)^3};$$

$$\therefore\ k\frac{ds}{dx} = -\frac{a\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^2};$$

$$\therefore\ ks + \text{const.} = a.\frac{1}{\frac{dy}{dx}} = a\frac{dx}{dy};\quad \text{also at } A,\ s = 0,\ \frac{dx}{dy} = 0;$$

$$\therefore\ \text{const.} = 0;$$

$$\therefore\ \frac{ks}{a} = \frac{dx}{dy};$$

which coincides with the equation to the common catenary when the origin is placed at the lowest point and x taken vertical. Hence this is the same curve as when the force is parallel and constant*.

* Soon after the time (1691) when the Problem of the figure of a chain acted upon by gravity was proposed and solved by the Bernoullis and Leibnitz, the attention of these geometers was directed to other curves which flexible bodies may assume under various circumstances. In particular the action of a fluid, whether by elasticity, weight, or impact, was considered; and as this action must be perpendicular to the

70. PROB. XV. *AP is acted upon by forces which are every where perpendicular to the curve, and which are, at every point P, proportional to the distance PE of P from a given line BE; to find the curve.*

Let BE be perpendicular to AB, $AB = c$; $PE = x$; $k =$ the force at A;

$$\therefore \frac{kx}{c} = -a \frac{\frac{d^2y}{dx^2}}{\left(\frac{ds}{dx}\right)^3};$$

which coincides with the equation to the elastic curve, as will be seen in the next Chapter, where that curve is considered.

We might now proceed to consider more complicated cases, as for instance when the flexible string rests upon any

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surface on which it acts, this case comes under Art. 68. of the text. One of their problems was, To find the figure of a rectangular *sail*, with two opposite sides fixed, inflated by the wind: and as the figure of a chain or cord had been called the *Catenaria* or *Funicularia*, this was called the *Velaria*. The weight of the sail itself being neglected, the problem may be solved on either of the following hypotheses:

1st, That the air which immediately presses the sail is, relatively to the sail, at rest; and of course kept in its place by the pressure produced by the wind behind. On this supposition it is the elasticity of the air which acts upon the curve; and since this force is the same at every point, the radius of curvature will be constant, and the curve will be a circular arc; consequently the surface will be a portion of a common cylinder.

2nd, That the air acts by impact, and produces no effect by pressure after the first impulse. This may be nearly the case when a single thread is stretched by a current of fluid, which can after the impact escape past it. In this case the force is as the square of the sine of the angle of impact, as appears from hydrodynamical principles. Hence this is the case of Prob. XIV, of the text, in which as is shewn, the curve is the common *Catenaria.*

It appears to have been supposed that the actual curve of the sail would be something compounded of both these forms.

Another problem of the same kind was, To find the form of a rectangle of cloth, &c. which having two opposite sides supported parallel to the horizon, is pressed by the weight of a fluid which is contained in it, and of course supposed to be prevented from running out at the ends. The curve of this problem was called the *Lintearia;* if BC, fig. 122, be the surface of the fluid, the pressure on any point P will be as the depth EP; hence the curve is the one found in Prob. XV; which, as is mentioned in the text, is the same with the *Elastica.*

curve surface or surfaces. We might also investigate the conditions of equilibrium of a flexible *surface* acted upon by gravity or by any forces. The mechanical principles of such problems would not present much difficulty after what has preceded, but the analytical results to which they would lead would in most cases be too complicated for an elementary work like the present.

On Suspension Bridges.

71. The curve formed by the chains or cords by which the road-way of a suspension bridge is supported will be a catenary if the weight supported by each part of the chain (namely the suspension rods, road-way, &c.) be proportional to the length of that part of the chain. We shall first consider the case of a suspension bridge on this supposition.

The tenacity of iron is such that a rod of 1 inch section will support the weight 14800 feet of the same rod; and the same is true for any other section: hence 14800 feet is the *modulus of tenacity* of this substance, and in like manner the tenacity of any other substance may be expressed by the length of the rod of the material of uniform thickness which the tenacity will support; and this length is the modulus of tenacity for that material.

The tension of the catenary at its vertex is represented by a length c, (Art. 46.) of the chain or cord, such that the weight of this length is equal to the force of the tension. And if the chain be loaded with additional weights distributed uniformly along its length, the tension at the vertex and at every other point will be increased in a constant proportion, that is, in the proportion of the augmented weight of any part.

72. PROP. *Given the width of a suspension bridge, to find its dimensions, so that the chains shall nowhere be loaded with more than a given fraction* (n) *of the weight they are able to sustain.*

Let c be the tension at the vertex, supposing the chain to support only its own weight; x the vertical abscissa from the vertex, y the horizontal ordinate: then the tension at any point is $x+c$. But if the weight of the chain be increased at every point (by the rods, road, &c.) in a constant ratio $1+m:1$, the tension at the vertex will be $(1+m)c$, and at any other point $(1+m)(x+c)$. Let l be the modulus of tenacity of the substance; the rods, road, &c. are supposed to add nothing to the tenacity of the chains. Hence l is the ultimate limit which the tension can attain, and nl is the limit which is prescribed in the proposition. Therefore

$(1+m)(x+c)$ must be less than nl.

Now y is given, being the half width of the bridge. And (Art. 48.)

$$x+c=\frac{c}{2}\left(\epsilon^{\frac{y}{c}}+\epsilon^{-\frac{y}{c}}\right).$$

Let N be the number of which $\frac{y}{c}$ is the natural logarithm,

$$\text{and } \epsilon^{\frac{y}{c}}=N \text{ .... (1.)}$$

Hence

$$x=\frac{c}{2}\left(N+\frac{1}{N}\right)-c=c\frac{(N-1)^2}{2N} \text{......(2).}$$

A simple mode of finding x from this is the following.

Assume $y=100$, and c equal to various values, at convenient intervals from 0 to 1000, or further if necessary.

Construct by formulæ (1) and (2) a Table of the values of x corresponding to each value of c; and of the corresponding values of the tension t at the extremity of the arc, which is always $x+c$.

The strength nl being expressed in the same scale in which y is 100, we shall find in the Table the greatest value of $(1+m)(x+c)$ which is less than nl; and this gives the greatest allowable value of x, the depth of the vertex of the curve below the points of suspension.

If a be the angle which the curve at the points of suspension makes with the vertical; by Art. 48,

$$\text{cotan. } a = \frac{dx}{dy} = \frac{1}{2}\left(\epsilon^{\frac{y}{c}} - \epsilon^{-\frac{y}{c}}\right) = \frac{N^2 - 1}{2N}.$$

And hence a may be found, and inserted in the same Table.

73. A Table such as we have described is inserted in the *Phil. Trans.* for 1826, p. 213, by Mr Davies Gilbert. The following is an extract from it. In addition to the quantities mentioned above, s is the length of the chain from the lowest point to the extremity.

$y = 100$

C	N	x	s	t	a
1000	1.11	5.00	100.16	1005.00	84° 16′
900	1.12	5.56	100.21	905.56	83 38
800	1.13	6.26	100.26	806.26	82 51
700	1.15	7.15	100.34	707.15	81 50
600	1.18	8.35	100.46	608.35	80 30
500	1.22	10.03	100.67	510.03	78 37
400	1.28	12.57	101.04	412.57	75 49
300	1.39	16.82	101.86	316.82	71 15
200	1.65	25.52	104.22	225.52	62 28
100	2.72	54.31	117.52	154.31	40 24

74. Thus let it be proposed to construct a bridge of 800 feet span, and let the adjunct weight of suspension rods, road-way, &c. be taken at one-half the weight of the chains: also let it be determined to load the chains with one-sixth of the breaking weight: to find the dimensions of the bridge.

The semi-span is 400 feet; hence the units of which y is 100 are 4 feet each; and l, the modulus of tenacity is $\frac{1}{4} \times 14800$ units, or 3700. Also m is $\frac{1}{8}$ and n is $\frac{1}{6}$. Hence $(1+m)(x+c)=nl$ gives $x+c=\frac{1}{9}\,l=411.125$.

Now in the Table we find that the tension nearest in value to this is 412.57, which corresponds to

$$c = 400 \text{ and to } x = 12.57;$$

that is, in feet,

$$c = 1600, \qquad x = 50.$$

Also the angle which the chain makes with the horizon at the points of suspension is $90^0 - 75^0\,49'$, or $14^0\,11'$.

It would appear by such a Table that for a given span, the tension at the points of suspension is least when $x = \frac{1}{8}$ the whole span nearly.

In the preceding reasoning, the weight of any portion of a suspension bridge is supposed to be proportional to the corresponding length of the suspended chain. This however is not exact, and we shall now consider the question without introducing this supposition.

75. In the chain bridge, the strain proceeds from three causes; the weight of the suspended chain;—the weight of the road-way;—and the weight of the suspension rods which connect the former two together by means of vertical lines. The last of these weights will generally be small compared with the others.

In this case we shall still have as before, (fig. 118),

$$\frac{\text{tension at } C}{\text{weight of } CP} = \frac{dy}{dx}; \text{ whence weight of } CP = c\frac{dx}{dy},$$

including in the weight of CP, the three portions we have mentioned.

76. Prop. *To find the general equation to the curve of a suspension bridge.*

Let the weight of a unit of length of the curve be m.

Let the weight of a unit of length of the road-way be n; the road-way being supposed to be horizontal and of uniform weight for different equal portions of its length.

And let the weight of a unit of vertical surface of the suspending rods be r; the rods being supposed to be uniformly distributed, and very near each other; and therefore being reckoned as a vertical surface.

Let x, y, s be taken as before, Art. 46. If we take δy a small portion of the horizontal ordinate, and suppose δs to be the corresponding portion of the curve, $m\delta s$ is the weight of the portion of the curve, and $m\delta y$ the weight of the portion of the road-way. Also $rx\delta y$ is the weight of the corresponding portion of the rods. Hence the whole weight corresponding to the element δs is

$$m\delta s + n\delta y + rx\delta y$$

$$\text{or } \left\{m + n\frac{\delta y}{\delta s} + rn\frac{\delta y}{\delta s}\right\}\delta s.$$

When we suppose the curve to be continuous, we must suppose δs and δy to be indefinitely small; in which case the ratios of such quantities are the differential coefficients. Hence the differential coefficient of the weight is

$$m + n\frac{dy}{ds} + rx\frac{dy}{ds},$$

and the whole weight is the integral of this, taken with regard to s; that is, it is

$$ms + ny + r\int_s x\frac{dy}{ds};$$

the integral being supposed to begin when $x = 0$ and $y = 0$.

Hence the equation above stated becomes

$$ms + ny + r\int_s x\frac{dy}{ds} = c\frac{dx}{dy}.$$

The road-way is here supposed to be divided by transverse cuts into indefinitely small separate parts, which is allowable, since the curve must always adjust itself so as to have the road-way a straight line.

77. Prop. *To find the nature of the curve when the weight of the suspension rods is neglected.*

In this case we make $r = 0$; and the equation is

$$ms + ny = c\frac{dx}{dy}; \quad \text{let } \frac{dx}{dy} = p.$$

$$ms + ny = cp;$$

$$c\frac{dp}{dy} = m\frac{ds}{dy} + n = m\sqrt{(1+p^2)} + n;$$

$$\frac{cp\dfrac{dp}{dx}}{m\sqrt{(1+p^2)}+n} = 1; \quad \text{assume } 1 + p^2 = q^2.$$

$$\frac{q\dfrac{dq}{dx}}{q+\dfrac{n}{m}} = \frac{m}{c}; \quad \text{whence } q - \frac{n}{m}\,\mathrm{l}\left(q + \frac{n}{m}\right) = \frac{mx}{c} + C;$$

$$\text{when } x = 0,\ p = 0,\ q = 1; \quad 1 - \frac{n}{m}\,\mathrm{l}\left(1 + \frac{n}{m}\right) = C.$$

$$q - 1 - \frac{n}{m}\,\mathrm{l}\frac{q + \dfrac{n}{m}}{1 + \dfrac{n}{m}} = \frac{mx}{c},$$

$$\text{or } \sqrt{(p^2+1)} - 1 - \frac{n}{m} \,\mathrm{l}\, \frac{\sqrt{(1+p^2)} + \dfrac{n}{m}}{1 + \dfrac{n}{m}} = \frac{mx}{c}.$$

Also

$$\frac{dy}{dx} = \frac{dy}{dp}\,\frac{dp}{dx} = \frac{c}{m\sqrt{(1+p^2)}+n} \cdot \frac{dp}{dx},$$

which may be rationalised by putting $\sqrt{(1+p^2)} = pq + 1$. We shall thus find

$$\frac{dy}{dq} = \frac{2c\,(1+q^2)}{(1-q^2)\,\{m+n-(n-m)\,q^2\}},$$

which may be integrated, because it is a rational fraction. The result will involve logarithms so long as n is greater than m.

If n be less than m, or the weight of a given length of road-way with its load be less than the weight of the same length of the chain, the logarithmic expression for y becomes imaginary, and the real integral will involve circular arcs. The reader will find this subject further discussed in the *Manchester Memoirs*, Vol. v. New Series; by Mr Hodgkinson, from whose paper the above investigation is taken.

CHAP. VI.

THE EQUILIBRIUM OF AN ELASTIC BODY.

78. BODIES are said to be *elastic* when they admit of a certain change of figure and dimensions, but possess a force which resists this change, which makes it depend upon the power applied, and which restores the bodies to their original dimensions and figure, when the power which altered them is removed. This restitutive energy acts in various ways.

1. *The Elasticity of Extension and Compression.*

A string may be stretched by a force applied lengthways to it, and an elastic surface or solid may be considered as a collection of elastic fibres.

It is found by experiment, that when a string is stretched, the increase of length is proportional to the force which produces it; that is, the *extension is as the tension**. We may also suppose the same law to extend to compression; but in order that a string may be susceptible of compression lengthways, it must be supposed to be inflexible.

2. *The Elasticity of Flexure.*

Wires and laminæ of different metals and other substances exert a force to unbend themselves when forcibly bent. In the flexure of elastic rods and laminæ, it appears by experiment that the deflexion, and consequently the curvature, is nearly as the force †. This also follows from

* See s'Gravesande's *Elem. Physices*, Lib. i. c. 26.

† See Boit's *Traité de Physique*, Tom. I. p. 509.

supposing an elastic rod to be composed of fibres which have elasticity of extension, as will be seen.

3. *The Elasticity of Torsion.*

Threads of metal, &c. when twisted, exert a force to untwist themselves. It appears from experiment*, that when very fine threads of metal are twisted by means of levers transverse to them, the force by which they tend to resume their natural state is very accurately as the angle of torsion.

1. *Elasticity of Extension.*

79. Prop. *When an elastic string of given length is stretched by a given force, to find its length.*

In a given elastic string the length added is, as we have said, proportional to the tension. If the tension be the same, the added length will, in different lengths of the same string, be proportional to the length; for it is manifest that a string two feet long will be twice as much extended by the same tension as a string one foot long; since the tension will be the same throughout, and therefore each of the halves of the first string will be as much stretched as the second string. In strings which differ in material, thickness, &c. the extension for a given length and tension, will be different for different substances; and will in each be proportional to a certain quantity which may be considered as the measure of the *extensibility* of the particular substance which is to be taken. If ϵ be this quantity for a certain string whose length at first (that is, when not stretched by any force) is a, when this string is stretched by a force or weight t, which will of course measure the tension, its increase of length will be proportional to $a\epsilon t$, and may be *equal* to this expression by properly assuming ϵ. Hence the length under these circumstances will be $a + a\epsilon t$, or $a(1 + \epsilon t)$.

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* For the experiments of Coulomb, see Biot, *Traité de Physique*, Tom. I. p. 492.

We may determine ϵ if we know the original length of the string and its length for any given value of t. It may be convenient to know it in terms of the force which will draw out the string to *double* its length. Let E be this force; hence

$$a(1+\epsilon E)=2a; \quad \therefore \epsilon E=1, \text{ and } \epsilon=\frac{1}{E}.$$

Hence the length of the string under a tension t becomes

$$=a\left(1+\frac{t}{E}\right).$$

E may be expressed by a length of the given string whose weight would draw the string a to double its length. E is then called the *Modulus of Elasticity*.

If the tension be not the same throughout the string, this formula is not applicable. In this case we may suppose the string divided into indefinitely small portions; and in each of these portions the tension may be supposed constant, and the extension of that part found; and by combining all these, we get the extension of the whole.

80. Knowing thus the relation of the length and tension of such lines, we can easily express the conditions required by the solution of problems in which they occur, as will appear by the following examples.

PROB. I. *Fig.* 30. *AC, BC, are two given equal and similar elastic strings fixed at two points A, B, in the same horizontal line, and supporting at C a weight W: knowing the extensibility of the strings, to find where W will be supported; the strings themselves being supposed without weight.*

It is manifest that the vertical line CE will bisect AB. Let $AE=b$, angle $CAE=\alpha$, weight at $W=w$, tension of AC or $BC=t$, extensibility of $AC=\epsilon$, original length of $AC=a$, hence $AC=a(1+\epsilon t)$.

Since W is supported by the tensions of AC, BC, in those directions, we have

$$w = 2t \sin. \alpha; \quad \text{also} \quad AE = AC \,.\, \cos. \alpha, \quad \text{or}$$

$$b = a\,(1 + \epsilon t) \cos. \alpha.$$

Eliminating t,

$$\frac{b}{a} = \cos. \alpha + \frac{\epsilon w}{2} \cot. \alpha \,\ldots\ldots\ldots (1).$$

If we should attempt to obtain α from this equation, we should arrive at an equation of four dimensions; and by solving this, we should find the position of equilibrium. But for the most common case, that is, when the extensibility is small, and the weight w not very large, we may easily deduce from our equation an approximation to the situation. For we have

$$\alpha = A + A'\epsilon + A'' \,.\, \frac{\epsilon^2}{1 \,.\, 2} + \ldots\ldots$$

when A, A', A'', ... are the values which α, $\frac{d\alpha}{d\epsilon}$, $\frac{d^2\alpha}{d\epsilon^2}$, ... assume by making $\epsilon = 0$, (Lacroix, *Elem. Treat.* Art. 21.)

Hence, putting 0 for ϵ in the fundamental equation, (1) and in its differentials, we obtain

$$\frac{b}{a} = \cos. A;$$

$$0 = -\sin. \alpha \frac{d\alpha}{d\epsilon} - \frac{\epsilon w}{2} \frac{1}{\sin.^2 \alpha} \frac{d\alpha}{d\epsilon} + \frac{w}{2} \cot. \alpha \frac{d\epsilon}{d\alpha};$$

$$\therefore\; A' = \frac{w}{2} \,.\, \frac{\cot. A}{\sin. A} = \frac{w}{2} \,.\, \frac{ab}{a^2 - b^2}, \quad \&c\ldots\ldots$$

Therefore

$$\alpha = A + \frac{w\epsilon}{2} \,.\, \frac{ab}{a^2 - b^2} + \&c\ldots\ldots$$

Here A is the angle BAC on the supposition that the strings

were inextensible: hence $\frac{w\epsilon}{2} \cdot \frac{ab}{a^2 - b^2}$ is, when ϵ is small, very nearly the quantity by which this angle is increased by supposing the strings extensible.

COR. 1. If $a = b$, that is, if the string ACB be just equal to AB when not stretched, we have from (1)

$$1 = \cos. \alpha + \frac{\epsilon w}{2} . \cot \alpha; \text{ and multiplying by } \tan. \alpha,$$

$$\tan. \alpha - \sin. \alpha = \frac{\epsilon w}{2}; \text{ and expanding } \tan. \alpha = \sin. \alpha \, (1 - \sin.^2 \alpha)^{-\frac{1}{2}},$$

$$\frac{1}{2} \sin.^3 \alpha + \frac{1 \, . \, 3}{2 \, . \, 4} \sin.^5 \alpha + \ldots . = \frac{\epsilon w}{2}.$$

If ϵ be small, α will be small; hence, neglecting the higher powers of sin. α,

$$\sin.^3 \alpha = \epsilon w; \quad \sin. \alpha = \sqrt[3]{(\epsilon w)} = \sqrt[3]{\frac{w}{E}}.$$

E being the tension which would double the string. Hence for the same string, fixed horizontally and not stretched, the small deflexion produced by a weight hung at the middle point is as the cube root of the weight.

COR. 2. If $a < b$, the string would not reach from A to B horizontally without being stretched.

In this case, the equation becomes, multiplying by tan. α,

$$\frac{b}{a} \tan. \alpha - \sin. \alpha = \frac{\epsilon w}{2}.$$

And when α is small, neglecting its higher powers, we may put α both for its sine and tangent; hence

$$\frac{b - a}{a} . \alpha = \frac{\epsilon w}{2} = \frac{w}{2E}; \quad \alpha = \frac{w}{2E} \cdot \frac{a}{b - a}.$$

Therefore in this case the deflexion varies as the weight w, if it be supposed small.

81. Prob. II. *A uniform elastic string hangs vertically, stretched by its own weight: to find its length.*

Let ϵ, as before, be its extensibility when its weight is not supposed to act. Let a be its length when it is supposed not stretched; and x the distance, on the same supposition, of any small element δx from the upper extremity, by which it is suspended. The part below the element δx is $a - x$, when it is not stretched; and as the quantity of matter is not altered by extension, the weight of this part when stretched is as $a - x$; and may be represented by $a - x$, if we represent weights by the corresponding lengths of the unstretched string. Hence the element δx will become

$$\delta x \left\{1 + \epsilon (a - x)\right\};$$

or if z be the distance from the upper extremity to a point whose distance in the unstretched state was x,

$$\frac{dz}{dx} = 1 + \epsilon (a - x);$$

$$\therefore z = x - \frac{\epsilon (a - x)^2}{2} + \text{constant};$$

and at the upper extremity where $x = 0$, $z = 0$;

$$\therefore z = x + \frac{\epsilon (2ax - x^2)}{2}.$$

At the lower extremity, $x = a$; let the stretched length $= l$;

$$\therefore l = a + \frac{\epsilon a^2}{2}.$$

Hence, $\frac{\epsilon a^2}{2}$ is the quantity by which the length of the string is increased when it is hung up. If E be a length of the string whose weight alone would be sufficient to stretch any part to twice its length, $\epsilon = \frac{1}{E}$, and $\frac{a^2}{2E}$ is the increment of length.

COR. 1. If we had $a = E$, we should have the length when stretched $= E + \frac{E^2}{2E} = \frac{3E}{2}$.

COR. 2. Since $l = a\left(1 + \frac{\epsilon a}{2}\right)$; it appears that the weight of the string stretches it half as much as if it were all collected at the lowest point.

82. PROB. III. *To find the catenary when the chain is extensible.*

Let the chain or cord be of uniform thickness and density, and let, as before, the elasticity be such that a length a becomes $a(1 + \epsilon t)$ by a tension t.

Let C, fig. 123, be the lowest point; and let the tension at C be equal to the weight of a length $CA = c$ of the unstretched string: $AN = x$, $NP = y$ the horizontal and vertical co-ordinates: $s =$ the arc CP, and $s' =$ the length of CP before it was stretched, which may therefore represent the weight of CP; $t =$ the tension at P.

If δs, $\delta s'$ be corresponding elements of s, s', we have

$$\delta s = \delta s'\ (1 + \epsilon)t; \quad \therefore \frac{ds'}{ds} = \frac{1}{1 + \epsilon t}.$$

The forces which keep CP at rest are the tension t at P, the tension c at C, and the weight s'. Hence these forces are as the sides of a triangle which are parallel to them; for instance, the elementary triangle at P, whose sides would be the elements δx, δy, δs: hence

$$\frac{t}{c} = \frac{ds}{dx}; \quad \frac{s'}{c} = \frac{dy}{dx};$$

By the second of these equations,

$$\frac{d^2y}{dx^2} = \frac{1}{c}\frac{ds'}{dx} = \frac{1}{c}\frac{ds}{dx}\frac{ds'}{ds} = \frac{\frac{ds}{dx}}{c + c\epsilon t};$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\frac{ds}{dx}}{c + c^2\epsilon\frac{ds}{dx}} \text{ by the first.}$$

If we make $\frac{dy}{dx} = p$, $\frac{ds}{dx} = \sqrt{(1+p^2)}$; and supposing the differentiation to be performed with respect to x, our last equation becomes

$$\frac{dp}{dx} = \frac{\sqrt{(1+p^2)}}{c + c^2\epsilon\sqrt{(1+p^2)}};$$

$$\therefore \frac{dx}{dp} = \frac{c}{\sqrt{(1+p^2)}} + c^2\epsilon;$$

$$\text{and } \frac{dy}{dp} = \frac{dy}{dx}\frac{dx}{dp} = p\frac{dx}{dp} = \frac{cp}{\sqrt{(1+p^2)}} + c^2\epsilon p.$$

Integrating these equations in p, we obtain

$$x = c\,\mathrm{l}\,\{p + \sqrt{(1+p^2)}\} + c^2\epsilon p;$$

$$y = c\sqrt{(1+p^2)} + \tfrac{1}{2}c^2\epsilon p^2;$$

the integrals being taken so that at C, where $p = 0$, we may have $x = 0$, and $y = c$.

By eliminating p, we should have the relation between x and y: and p is the tangent of the angle which the curve at P makes with the horizon.

$$\frac{ds}{dx} = \sqrt{(1+p^2)} = c\frac{dp}{dx} + c^2\epsilon\sqrt{(1+p^2)}\cdot\frac{dp}{dx};$$

$$\therefore s = cp + \tfrac{1}{2}c^2\epsilon\,\{p\sqrt{(1+p^2)} + \mathrm{l}\,[p + \sqrt{(1+p^2)}]\},$$

$$t = c\frac{ds}{dx} = c\sqrt{(1+p^2)}.$$

It appears that the values of x, y, and s, consist of two parts; namely, terms independent of ϵ, which are the same as they would be in a cord not extensible; and terms which

involve ϵ. Hence if CP and CP' be arcs of an extensible and of an inextensible catenary, for which the value of c, that is, the tension at C, is the same; and the values of p the same, that is, the tangents, PT, $P'T'$ parallel; $P'O$ and OP being horizontal and vertical, we have

$$P'O = c^2\epsilon p, \quad OP = \tfrac{1}{2}c^2\epsilon p^2.$$

The tension t is the same in both cases, and CP' is the length of CP not stretched.

COR. If PT meet OP' in Q, $OQ = \dfrac{OP}{p} = \tfrac{1}{2}c^2\epsilon p = \tfrac{1}{2}OP'$.

From these few examples it will be seen how problems involving extensible lines may be reduced to calculation.

2. *Elasticity and Resistance of Solid Materials.*

83. All solid substances, as wood, stone, metals, &c., are susceptible of some compression and extension. This compression and extension are greater as the forces producing them are greater; and when the forces produce a compression or extension greater than the texture of the substance can bear, the bodies are crushed or broken. We shall here find the change of figure of such bodies when they are compressed under given circumstances.

We shall suppose that all solid bodies may be considered as made up of elastic fibres, capable of extension and compression. We shall also suppose, as in the last Section, that the resistance to extension is proportional to the extension in each fibre, and the same of compression. We shall further assume, that the resistance to extension and to compression are the same in the same fibre.

These principles would follow if we were to suppose the particles of bodies to be kept in equilibrium by their mutual forces in the natural state of the body; and the change to be small, which they undergo by the action of any force. In this case it might be proved that the displacement of a

given particle would be ultimately as the force which produces it.

When a solid body is acted on by any force, it may be partly extended and partly compressed. Thus let a mass $ABQP$, fig. 124, be acted upon by a force F, compressing it in the direction EF. The surface PNQ may be brought into the direction pNq; in this case all the fibres RR' which are on one side of N are shortened; all those on the other side of N are lengthened. NN' remains the same as in the natural state. N is called the *neutral point*, and the line which separates the parts of a transverse section of the body which are compressed, from those which are elongated is called the *neutral line* of that section.

84. PROP. *When a rectangular prismatic mass is compressed by a force parallel to the direction of the axis; to find the neutral line.*

Let AB, fig. 124, be the rectangular base of the mass, MM' its axis. And let the slice $UTPQ$ be compressed so as to assume the form $UTpq$, N being the neutral line. Then any fibre parallel to the axis, as VR, is compressed so that its length becomes Vr: and by the supposition, if t be the force compressing it, E the modulus of elasticity, as in last Article; we shall have

$$Rr = VR\,\frac{t}{E}; \quad \text{and hence } t = E\,\frac{Rr}{VR}.$$

Let $PM = MQ = a$, $MF = h$, $MR = x$, and the breadth of the beam perpendicular to $AB = b$; $MN = n$, whence $RN = n + x$; force at $F = f$.

Also let UT and QP meet in O, and let $OK = \rho$.

Hence

$$\frac{Rr}{VR} = \frac{Rr}{NL} = \frac{NR}{OL} = \frac{n+x}{n+\rho}.$$

And the force of VR, supposing its breadth and thickness each 1, is

$$t = E \cdot \frac{Rr}{VR} = E \cdot \frac{n + x}{n + \rho}.$$

Hence if we take a very thin portion, of which the thickness is δx and breadth b, its force is

$$E \cdot \frac{n + x}{n + \rho} \cdot b\delta x,$$

and this is the increment of the force exerted at R corresponding to δx. When x is negative and greater than n, this is negative; and accordingly the compression for that part becomes extension.

The forces which keep each other in equilibrium are the force f acting at F, and the elementary forces of all the fibres VR. And hence, by Art. 24, we must have, 1st, the force f equal to all the forces

$$E \cdot \frac{n + x}{n + \rho} b\delta x;$$

and 2nd, the moment of the force f about N equal to the moments of all the forces

$$E \cdot \frac{n + x}{n + \rho} b\delta x \text{ about } N.$$

Also the aggregate of all the forces will be found by taking the coefficients of δx, in the expressions so found, and the integrals of these differential coefficients from

$$x = -a, \text{ to } x = a.$$

Hence we have

$$f = \textstyle\int_x E \, \dfrac{n + x}{\rho + n},$$

$$f(h + n) = \textstyle\int_x E \, \dfrac{(n + x)^2}{\rho + n}.$$

Integrating between the proper limits,

$$f = E \cdot \frac{2nab}{\rho + n},$$

$$f(h+n) = E \cdot \frac{2n^2ab + \frac{2}{3}a^3b}{\rho + n}.$$

Dividing, we have

$$h + n = n + \frac{a^2}{3n};$$

$$\therefore n = \frac{a^2}{3h} = \frac{(2a)^2}{12h}. \quad \text{And } MN = \frac{PQ^2}{12MF}.$$

Cor. 1. If $MF = \frac{1}{3}MP$, or $h = \frac{1}{3}a$, $n = a$, the neutral point is in the surface, and the whole beam is compressed.

If $MF > \frac{1}{3}MP$, the neutral point is beyond the surface.

Cor. 2. From the above equations we have

$$\rho + n = \frac{E}{f} \cdot 2nab = \frac{E}{f} \cdot \frac{2a^3b}{3h}.$$

And $\rho + n$ is the radius of curvature of the neutral line NN' at N. Let the force f be equivalent to a length F of the prism; then $f = 2Fab$; and we have

$$\rho + n = \frac{E}{F}\frac{a^2}{3h}; \quad \text{or } NO = \frac{E}{F} \cdot \frac{PQ^2}{12MF}.$$

85. Prop. *When a rectangular prism is acted upon by any force in any direction; to find the neutral point at any part.*

Let a force f, fig. 124, act in the line yF on a prism $ABPQ$. The force will produce the same effect as if acted at F, a point in QP. Let the angle MFy at $F = \alpha$. The force may be resolved into $f \cos. \alpha$ in QP, and $f \sin. \alpha$ perpendicular to QP. Of these the former is resisted by the lateral cohesion of the materials, and produces no compression. The latter produces a compression as in last Article. Hence, retaining the denominations of last Article, calling MF, h, and putting $f \sin. \alpha$ for f, we have

$$f \sin. a = E . \frac{2nab}{\rho + n};$$

$$f(h+n) \sin. a = E . \frac{2n^2ab + \frac{2}{3}a^3b}{\rho + n}.$$

And hence $h + n = n + \dfrac{a^2}{3h}$; $\therefore\ n = \dfrac{a^2}{3h}$.

Cor. 1. We have also

$$\rho + n = \frac{E}{f \sin. a} . 2nab = \frac{E}{f \sin. a} . \frac{2a^3b}{3h}$$

$$= \frac{E}{f} . \frac{2a^3b}{3h \sin. a} = \frac{2Ea^3b}{3fk},$$

if $k = My = h \sin. a$, the perpendicular on the direction of the force from the axis.

Cor. 2. If as before $f = 2Fab$,

$$\text{rad. of curv.} = \rho + n = \frac{Ea^2}{3Fk}.$$

Cor. 3. If the force act perpendicularly to the axis, h is infinite, $n = 0$, and the neutral point is in the axis.

86. Prop. *When a rectangular prismatic beam is made to deviate a little from a straight line by the action of a given force perpendicular to it, to find the deflexion.*

Since the force is perpendicular to the beam, and the beam is nearly a straight line, we may, by Cor. 3, of last Article, suppose the neutral point to be every where coincident with the axis. Let AME, fig. 125, represent the axis, bent by a force acting perpendicularly to AD, its original position. And let XM be the ordinate at any point, also perpendicular

to AD. $AX = x$, $XM = y$. And since the curve is nearly a straight line, $\frac{dy}{dx}$ is small: hence the radius of curvature

$$= \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \text{ is } = \frac{1}{\frac{d^2y}{dx^2}}, \text{ nearly.}$$

But by Cor. 2. to last Art. if $AD = l$, $k = DX = l - x$,

$$\text{rad. of curv.} = \frac{E}{F} \cdot \frac{a^2}{3(l-x)};$$

$$\therefore \frac{d^2y}{dx^2} = \frac{F}{E} \cdot \frac{3(l-x)}{a^2}.$$

Integrate with respect to x, observing that $\frac{dy}{dx} = 0$ when $x = 0$,

$$\therefore \frac{dy}{dx} = \frac{F}{E} \cdot \frac{3lx - \frac{3}{2}x^2}{a^2}.$$

Integrate again: observing that $y = 0$, when $x = 0$.

$$y = \frac{F}{E} \cdot \frac{\frac{3}{2}lx^2 - \frac{1}{2}x^3}{a^2}.$$

And if the whole deflexion $DE = \delta$, making $x = l$,

$$\delta = \frac{F}{E} \cdot \frac{l^3}{a^3}.$$

Cor. 1. If we put for F its value $\frac{f}{2ab}$, we have

$$\delta = \frac{fl^3}{2Ea^3b}.$$

Hence it appears that for a given breadth and thickness the deflexion is as the force and cube of the length.

And for a given force and length the deflexion is inversely as the breadth and cube of the thickness.

Cor. 2. Let the direction of the tangent at E make an angle θ with the tangent at A. Then θ may be called the *angular deflexion.*

And $\frac{dy}{dx} = \tan.\theta$; hence, putting l for x in the value of $\frac{dy}{dx}$,

at the extremity, $\tan.\theta = \frac{F}{E} \cdot \frac{3l^2}{2a^2} = \frac{3fl^2}{4a^3b}$.

The extreme angular deflexion is as the force and square of the length.

87. Prop. *When a rectangular prismatic beam, fixed in a horizontal position, is bent by its own weight, (its thickness being vertical) to find the deflexion.*

In Art. 85, Cor. 2; put Fk the moment of the force which bends the beam $= (l - x)\frac{l - x}{2} = \frac{1}{2}(l - x)^2$; and for the rad. of curv. $\dfrac{1}{\frac{d^2y}{dx^2}}$.

Hence we have

$$\frac{d^2y}{dx^2} = \frac{3(l - x)^2}{2Ea^2}; \quad \frac{dy}{dx} = \frac{l^3 - (l - x)^3}{2Ea^2};$$

$$y = \frac{l^3x + \frac{1}{4}(l - x)^4 - \frac{1}{4}l^4}{2Ea^2};$$

and the whole deflexion $\delta = \frac{3l^4}{8Ea^4}$.

Cor. In this and the last Article, δ being observed, E may be found.

88. Prop. *When an isosceles triangular prism is acted upon by any force in any direction, to find the neutral point at any part.*

The force is supposed to act in the plane which bisects the vertical angle of the isosceles triangle. Let $ABQP$, fig. 124, be this plane, the vertex of the triangle being at P, and its base at Q.

Let $OT=\rho$, $TV=x$, $TL=n$, $TU=a$, $PF=h$, $MFy=a$, and the force $=f$, modulus of elasticity $=E$.

As before, in Art. 85, we shall have the force of a single fibre at $R = E\dfrac{NR}{OL} = E\dfrac{n-x}{\rho+n}$.

And whatever be the form of the section perpendicular to the plane $ABQP$, if y be the ordinate of this section perpendicular to the line PQ, we shall have for the elementary force exerted at R,

$$E\frac{n-x}{\rho+n}y\delta x.$$

And by the same reasoning as in Art. 85,

$$f\sin. a = E\int_x \frac{n-x}{\rho-n}y,$$

$$f(h+n)\sin. a = E\int_x \frac{(n-x)^2}{\rho+n}y.$$

In the case of the triangle, $y=mx$, m being a constant quantity. And integrating from $x=0$ to $x=a$,

$$f\sin. a = \frac{Em}{\rho+n}\cdot\left(\frac{1}{2}na^2 - \frac{1}{3}a^3\right),$$

$$f(h+n)\sin. a = \frac{Em}{\rho+n}\left(\frac{1}{2}n^2a^2 - \frac{2}{3}na^3 + \frac{a^4}{4}\right);$$

$$\therefore\ h + n = \frac{6n^2 - 8na + 3a^2}{6n - 4a}$$

$$= \frac{3a^2 - 4an}{6n - 4a} + n;$$

$$\therefore h = \frac{3a^2 - 4an}{6n - 4a};$$

$$\therefore n = \frac{3a^2 + 4ah}{4a + 6h}.$$

COR. 1. If $h = 0$, or the force act at P, $n = \frac{3}{4}a$.

COR. 2. If the force act perpendicularly to the prism, h is infinite, and $n = \frac{2a}{3}$.

COR. 3. If the force act above P, h will be negative. Thus if the force act at Q, $h = -a$, $n = \frac{a}{2}$.

COR. 4. To find the radius of curvature of the neutral line, we have

$$\text{rad. curv.} = \rho + n = \frac{Em}{f \sin. a}\left(\frac{1}{2}na^2 - \frac{1}{3}a^3\right);$$

and putting for n its value,

$$\text{rad. curv.} = \frac{Em}{f\sin. a} \cdot \frac{a^4}{6(4a + 6h)} = \frac{Ema^4}{36f\left(h + \frac{2a}{3}\right)\sin. a}.$$

And if we take a point distant from P by $\frac{2}{3}$ PQ, and from this point draw a perpendicular on the line of direction of the force; if this perpendicular $= k$,

$$k = \left(h + \frac{2a}{3}\right)\sin. a; \quad \text{rad. curv.} = \rho + n = \frac{Ema^4}{36fk};$$

or if b be the base of the triangle, $ma = b$, $\rho + n = \frac{Ea^3b}{36fk}$.

Cor. 5. If f be the weight of a length F of the prism, $f=\frac{1}{2}Fab$;

$$\therefore\ \rho+n=\frac{Ea^2}{18Fk}.$$

In the same manner we might find the neutral point for prismatic beams of other figures. And the deflexion when they are acted on by given weights would be found in the same manner as before.

Also if the beams are not prismatic, a will be variable; and by putting for it the expression belonging to each case, we may find the deflexion in beams of other forms.

89. Prop. *A rectangular prismatic beam is compressed by a given force acting in a direction parallel to the axis; to find the deflexion.*

Let $ABA'B'$, fig. 126, be the beam, FF' the line in which the force acts. P any point in the axis. And since the deflexion is supposed to be small, PM, which is perpendicular to FF', may be considered as perpendicular also to the axis. Hence if a be half the thickness of the beam $(=\frac{1}{2}AB)$ and n the distance of the neutral point above P, $EM=x$, $PM=y$, we have, by Art. 85, $n=\dfrac{a^2}{3y}$.

Also if ρ be the radius of curvature of the axis CP, by Cor. 2, of the same Article,

$$\rho+n=\frac{E}{F}\cdot\frac{a^2}{3y};\quad\therefore\ \rho=\left\{\frac{E}{F}-1\right\}\frac{a^2}{3y}=\frac{c^2}{y},\ \text{suppose.}$$

Now $\dfrac{1}{\rho}=-\dfrac{d^2y}{dx^2}$ nearly, because the deflexion is small;

$$\therefore\ \frac{d^2y}{dx^2}=-\frac{y}{c^2}.$$

$$\text{Integrate, }\therefore\ \frac{dy^2}{dx^2}=C-\frac{y^2}{c^2}.$$

And if k be EV, the greatest ordinate, $y = k$ when $\dfrac{dy}{dx} = 0$;

$$\therefore \frac{dy^2}{dx^2} = \frac{k^2 - y^2}{c^2}; \quad \frac{1}{\sqrt{(k^2 - y^2)}} = -\frac{1}{c}\frac{dx}{dy};$$

$$\therefore \text{arc}\left(\cos. = \frac{y}{k}\right) = \frac{x}{c}; \quad x \text{ being measured from } E,$$

$$y = k \cos. \frac{x}{c}.$$

Let $l = EF =$ half the length of the beam. And let $h = CF$, the distance of the force from the axis. Therefore when $x = l$, $y = h$,

$$h = k \cos. \frac{l}{c}; \quad y = h . \frac{\cos. \frac{x}{c}}{\cos. \frac{l}{c}}.$$

Hence $EV = h \sec. \dfrac{l}{c}$; and DV the deflexion $= EV - FC$;

$$\therefore \text{deflexion} = h\left\{\sec. \frac{l}{c} - 1\right\}.$$

$$\text{But } c^2 = \frac{a^2}{3}\left\{\frac{E}{F} - 1\right\}; \quad \therefore \frac{l}{c} = \frac{l}{a}\frac{\sqrt{3F}}{\sqrt{E - F}}.$$

COR. 1. If E be very large compared with F, we shall have the deflexion

$$= h\left\{\sec. \frac{l\sqrt{3F}}{a\sqrt{E}} - 1\right\}.$$

COR. 2. The radius of curvature at V

$$= \frac{c^2}{k} = \frac{c^2 \cos. \frac{l}{c}}{h} = \frac{a^2 \cos. \frac{l}{c}}{3h}\left\{\frac{E}{F} - 1\right\}.$$

And when E is very large compared with F,

$$\text{rad. curv. at } V = \frac{Ea^2}{3Fh} \cos \frac{l\sqrt{3F}}{a\sqrt{E}}.$$

Cor. 3. The deflexion will be greater, as the secant, in Cor. 1, is greater; and when the secant is infinite, the formula will fail; in this case the prism will either be crushed, or will bend so much that the above reasoning is no longer applicable. And this will be the case if the arc be a quadrant. Hence in order that the prism may support a weight with a small deflexion, the weight acting on one side of the axis, we must have

$$\frac{l\sqrt{3F}}{a\sqrt{E}} < \frac{\pi}{2},$$

$$\frac{l^2}{a^2} < \frac{\pi^2 E}{12F}.$$

Cor. 4. If the force act at the extremities of the axis, $h = 0$; and there will be no deviation except the secant of the arc be infinite; that is, except

$$\frac{l^2}{a^2} = \frac{\pi^2 E}{12F} = .8225 \frac{E}{F}.$$

Hence we may find the weights which columns of given materials will support. Thus, if in fir-wood the modulus E be 10000000 feet, a bar an inch square and 10 feet long may begin to bend when

$$F = .8225 \times \left(\frac{1}{120}\right)^2 \times 10000000 = 571 \text{ feet};$$

that is, it will bend when pressed by the weight of 571 feet of the same bar, or about 120 pounds, neglecting the pressure arising from the weight of the bar itself.

The modulus of elasticity for iron or steel is about 9000000 feet; for wood, from 4000000 to 10000000; and for stone, probably about 5000000.

Cor. 5. In the same manner we might find the deflexion of a triangular prismatic beam acted on by a longitudinal force. For in this case, supposing E large with respect to F,

$$\rho = \frac{Ea^2}{18Fy}.$$

3. *The Curves formed by Elastic Laminæ.*

90. If we consider the thickness of the elastic bodies in Art. 85, to be small, we may neglect n, and we have, when the section of the body is a rectangle,

$$\rho = \frac{2Ea^3b}{3fk};$$

and in all cases $\rho = \frac{\text{E}}{fk}$; when E is a constant quantity depending upon the size and form of the section of the elastic body, and upon its elasticity. If we suppose the body to be a lamina of uniform thickness, the value of a will be constant, and E will be proportional to b.

Prop. 91. *An elastic lamina of uniform breadth and thickness is fixed at one end and acted upon by a given force; it is required to determine the form of the curve.*

Let BA, fig. 127, be the lamina, fixed at B; f the force, which acts at A or E in the direction AE; $CM = x$, $MP = y$, co-ordinates perpendicular and parallel to the direction of the force AE; $AP = s$. The radius of curvature at P is

$$-\frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Now it will manifestly make no alteration in the curvature at any point, as P, whether, after the equilibrium is established,

we suppose the part PA rigid or not, or of one form or another. Hence the force f may be supposed to act on a straight rigid arm $PK = x$; and we have by the last Article,

$$fk = \frac{\mathrm{E}}{\rho}, \text{ or } fx = -\frac{\mathrm{E}\dfrac{d^2y}{dx^2}}{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}} \dots\dots\dots (1).$$

If $\dfrac{dy}{dx} = p$, this becomes

$$fx = -\frac{\mathrm{E}\dfrac{dp}{dx}}{(1+p^2)^{\frac{3}{2}}};$$

and integrating,

$$\frac{f}{2}(b^2+x^2) = -\frac{\mathrm{E}p}{\sqrt{(1+p^2)}} \dots\dots\dots (2),$$

b^2 being an arbitrary constant, to be determined. Hence, obtaining p^2,

$$p^2 = \frac{f^2(b^2+x^2)^2}{4\mathrm{E}^2 - f^2(b^2+x^2)^2}; \text{ and, making } a^2 = \frac{2\mathrm{E}}{f},$$

$$p^2 = \frac{(b^2+x^2)^2}{a^4-(b^2+x^2)^2} = \frac{(b^2+x^2)^2}{(a^2-b^2-x^2)(a^2+b^2+x^2)};$$

$$\therefore \frac{dy}{dx} = \pm\frac{b^2+x^2}{\sqrt{(a^2-b^2-x^2)(a^2+b^2+x^2)}};$$

$$\text{also } \frac{ds}{dx} = \sqrt{(1+p^2)} = \pm\frac{a^2}{\sqrt{(a^2-b^2-x^2)(a^2+b^2+x^2)}}.$$

We must determine b^2 from known circumstances in the problem. If the curve BP be continued, to meet the line AE, and at the point of intersection make with the line of abscissas an angle α, we shall easily determine b^2. Since at that point $x = 0$ and $p = \tan.\alpha$, equation (2) becomes

$$\frac{fb^2}{2} = -\frac{\mathrm{E}\tan.\alpha}{\sec.\alpha}; \quad \therefore b^2 = -\frac{2\mathrm{E}\sin.\alpha}{f} = -a^2\sin.\alpha.$$

If the curve do not meet the line AE, b^2 must be otherwise determined, as will be seen hereafter.

92. Making $a^2 - b^2 = c^2$, whence $a^2 + b^2 = 2a^2 - c^2$, our equations become

$$\frac{dy}{dx} = \pm \frac{a^2 - c^2 + x^2}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}} \quad \ldots\ldots\ldots (3).$$

$$\frac{ds}{dx} = \pm \frac{a^2}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}} \quad \ldots\ldots\ldots (4).$$

When $x = 0$, as at A, the curve makes an angle α with the abscissa. When $a^2 - c^2 + x^2 = 0$, or $x^2 = c^2 - a^2 = -b^2 = a^2 \sin. \alpha$, or $x = a \sin.^{\frac{1}{2}} \alpha$, we have $\frac{dy}{dx} = 0$ and the curve is parallel to the abscissa.

When $x = c$, $\frac{dy}{dx}$ becomes infinite, and the curve is perpendicular to the axis. When x is greater than this, the expression is impossible. Hence $c = ED$. Beyond this point the curve turns back with an arc DC, fig. 127, similar to the arc AD before this point: and these two arcs correspond to the double sign of $\frac{dy}{dx}$ in (3).

If we find the radius of curvature we shall obtain it $= \frac{a^2}{2x}$. Hence the radius of curvature at the points A, C, A', &c. where $x = 0$, is infinite. These are points of contrary flexure, and the curve between each successive two of them consists of similar arcs placed alternately. The curve, as determined from the equation, may be continued indefinitely in this form.

93. To obtain the values of y and s we should have to integrate equations (3) and (4). The expressions, however,

cannot be integrated in finite terms*. We may easily integrate them in series, by making $\sqrt{(c^2 - x^2)} = u$; whence we have, neglecting the signs,

$$\frac{ds}{dx} = \frac{a^2}{u\sqrt{(2a^2 - u^2)}}, \quad \frac{dy}{dx} = \frac{a^2 - u^2}{u\sqrt{(2a^2 - u^2)}},$$

$$\frac{ds}{dx} - \frac{dy}{dx} = \frac{u}{\sqrt{(2a^2 - u^2)}}.$$

Expanding $\dfrac{1}{\sqrt{(2a^2 - u^2)}}$ by the binomial theorem, these equations become

$$\frac{ds}{dx} = \frac{1}{\sqrt{2}} \cdot \left\{\frac{a}{u} + \frac{1}{4} \cdot \frac{u}{a} + \frac{1.3}{4.8} \frac{u^3}{a^3} + \&c.\right\},$$

$$\frac{ds}{dx} - \frac{dy}{dx} = \frac{1}{\sqrt{2}} \cdot \left\{\frac{u}{a} + \frac{1}{4} \cdot \frac{u^3}{a^3} + \frac{1.3}{4.8} \cdot \frac{u^5}{a^5} + \&c.\right\}.$$

It is only necessary to take the integrals from $x = 0$, to $x = c$, which give AD and ED, fig. 127. Now, since

$$u = \sqrt{(c^2 - x^2)},$$

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* These expressions are of the kind which have been called *Elliptical Transcendentals*, from their connexion with the functions on which the rectification of elliptical arcs depends. Though the integration cannot be effected rigorously, many properties and relations of them have been discovered, and methods of finding the integrals within any requisite degree of approximation. The student will find these very completely treated of in the *Exercices de Calcul Integral* of Legendre; to whom, along with Euler and Lagrange, we are indebted for the discoveries made in this province of analysis.

If we make $x = c \,.\, \sin.\, \phi$, we shall find $s = \pm \frac{a}{\sqrt{2}} \cdot \int_\phi \frac{1}{\sqrt{(1 - m^2 \,.\, \sin.^2 \phi)}}$, putting $\frac{c^2}{2a^2} = m^2$: which is what Legendre calls an elliptical function of the first order, and designates by F. Similarly, y is reducible to elliptical functions. It appears from the work above-mentioned, that though we cannot find the length of an arc s, we can determine arcs double, treble, &c. or the halves, thirds, &c. of given arcs; with many other properties, for which the reader is referred to the work itself. We can also obtain very converging series for the integrals; both when m is small, (which we have given in the text,) and when m is nearly $= 1$; and likewise for other cases, in which the calculation is facilitated by the Tables given by Legendre.

we shall have, between these limits,

$$\int_x \frac{1}{u} = \int_x \frac{1}{\sqrt{(c^2 - x^2)}} = \frac{\pi}{2};$$

and by the known methods of finding $\int_x (c^2 - x^2)^{\frac{2n+1}{2}}$, (Lacroix, *Elem. Treat.* Art. 171.), we shall find between the same limits,

$$\int_x u = \frac{1}{2} \cdot \frac{\pi}{2} \cdot c^2;$$

$$\int_x u^3 = \frac{1.3}{2.4} \cdot \frac{\pi}{2} \cdot c^4; \text{ and so on.}$$

Hence if the length $ADC = l$, and the height $AC = h$;

$$\frac{l}{2} = \frac{\pi a}{2\sqrt{2}} \cdot \left\{1 + \frac{1^2}{2^2} \cdot \frac{c^2}{2a^2} + \frac{1^2.3^2}{2^2.4^2} \cdot \frac{c^4}{4a^4} + \ldots\right\}.$$

$$\frac{l-h}{2} = \frac{\pi a}{2\sqrt{2}} \cdot \left\{1 - \frac{1.c^2}{2a^2} + \frac{1^2.3}{2^2.4} \cdot \frac{c^4}{2a^4} + \frac{1^2.3^2.5}{2^2.4^2.6} \cdot \frac{c^6}{4a^6} + \ldots\right\};$$

$$\therefore\ l = \frac{\pi a}{\sqrt{2}} \cdot \left\{1 + \frac{1^2}{2^2} \cdot \frac{c^2}{2a^2} + \frac{1^2.3^2}{2^2.4^2} \cdot \frac{c^4}{4a^4} + \ldots\right\} \ldots\ldots\ldots (5)$$

$$h = \frac{\pi a}{\sqrt{2}} \cdot \left\{1 - \frac{1^2}{2^2} \cdot \frac{3}{1} \cdot \frac{c^2}{2a^2} - \frac{1^2.3^2}{2^2.4^2} \cdot \frac{5}{3} \cdot \frac{c^4}{4a^2} - \ldots\right\} \ldots (6)$$

and knowing

$$\frac{a}{\sqrt{2}} = \sqrt{\frac{E}{f}}, \text{ and } \frac{c^2}{2a^2} = \frac{a^2 - b^2}{2a^2} = \frac{1 + \sin\,\alpha}{2} = \cos.^2 \left(\tfrac{1}{4}\pi - \tfrac{1}{2}\alpha\right),$$

we may calculate l and h approximately.

From equation (3) we must determine the species of the curve. They will depend on the value of c compared with a.

94. Prop. *When the elasticity is variable, to determine the curve, having given the elasticity, and conversely.*

We have supposed the moments of the forces which tend to bend the lamina at any point to be equal to $\frac{\text{E}}{\rho}$, where E is the measure of the elasticity, and is the same for every point. We shall now suppose E to be a function of the curve or its co-ordinates. As before, let a force f act on the lamina and let the abscissa be perpendicular to the direction of the force. Hence

$$fx = \frac{\text{E}}{\rho}; \quad \therefore \text{E} = fx\rho.$$

If E be given in terms of s, or of x and y, we may substitute and integrate. If E be to be found, it will be had from the formula,

$$\text{E} = -\frac{fx\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

E, as appears from Art. 90, may be supposed proportional to the breadth, when the thickness is constant, and to the cube of the thickness, when the breadth is constant.

PROB. *To find how the breadth of a uniform elastic lamina must vary, that by a weight hung at the end of it, it may be bent into the form of a quadrant. Fig.* 128.

In this case ρ is constant; therefore E, $=f\rho x$, is as x. Hence the lamina must be such that its projection ACc on a horizontal plane is a triangle.

95. Hitherto we have supposed that the elastic rod or lamina in its natural state, when it is not acted on by any forces, is a straight line. But we may suppose that it is naturally of any form whatever, and that it is deflected from this natural form by the same laws by which we before supposed it deflected from a straight line.

PROP. *In an elastic rod which is naturally a given curve, the curvature produced by any force at any point*

is equal to the natural curvature, together with the curvature which the same force would produce in a rectilinear rod of the same elasticity, acting in the same manner.

Let Pq, fig. 129, be a small given arc whose natural curvature is Pq, and its center of curvature o; and let it be bent into the position PQ, with its center of curvature at O, by means of a force acting at the arm QE. Then the deflexion Qq, of Q from its natural position, is the same which it would be if Pq were a straight line.

Now ultimately, when PQ or Pq is indefinitely small, Qq may be considered as perpendicular to the tangent at P, and will therefore be equal to the difference of the perpendiculars QR and qr upon the tangent. Hence (Newt. *Prin.* Lem. XI.)

$$Qq = QR - qr = \frac{PQ^2}{2PO} - \frac{Pq^2}{2Po} = \frac{PQ^2}{2} \cdot \left\{\frac{1}{PO} - \frac{1}{Po}\right\}.$$

But if Pq were a straight line, Po would be infinite; and if $Q'q'$ be the deflexion in this case for an arc PQ', and PO' the radius of curvature for the same force;

$$Q'q' = \frac{PQ'^2}{2} \cdot \frac{1}{PO'}.$$

And by supposition the deflexion from the natural form is the same in the two cases for the same arc: or $Q'q' = Qq$, PQ' being equal to PQ. Hence

$$\frac{1}{PO} - \frac{1}{Po} = \frac{1}{PO'},$$

$$\text{and } \frac{1}{PO} = \frac{1}{PO'} + \frac{1}{Po};$$

and the curvature being inversely as the radius, the Proposition is manifest.

COR. Since, by Art. 90, $\frac{1}{PO'} = \frac{E}{fk}$,

we have $\dfrac{1}{PO} = \dfrac{fk}{\text{E}} + \dfrac{1}{Po}$.

E being, as before, a quantity which measures the elasticity of the rod PA; and fk the moment of the force which acts.

96. Prop. *A uniform elastic rod, which is naturally a given curve, is fixed at one end and acted on by a given force: it is required to find the form which it will assume.*

Let BA, fig. 127, be the curve when the force f is applied. And as before, CM perpendicular to $AE = x$, $MP = y$, $AP = s$. And let the radius of curvature of any point P be, in the original form, $= r$, and in the form which it assumes, $= \rho$. Hence

$$\frac{1}{\rho} = \frac{fx}{\text{E}} + \frac{1}{r}, \quad \text{or } fx = \frac{\text{E}}{\rho} - \frac{\text{E}}{r};$$

and r being given in terms of s, we have a differential equation to the curve AB.

97. Prob. *Fig.* 130. *The curve Ba, being originally a quadrant, fixed at its lowest point B, it is required to find the curve BA, when it is acted on by the force F.*

Let FA meet the horizontal line BD in D: $DM = x'$, $MP = y$; radius of $Ba = r$; and since the original curvature is in a direction contrary to that which the force would produce, r must be made negative in the formula. Hence it becomes

$$fx' = \frac{\text{E}}{\rho} + \frac{\text{E}}{r}; \quad \text{or if we make } x' - \frac{\text{E}}{fr} = x,$$

$$fx = \frac{\text{E}}{\rho}; \quad \text{and, putting for } \rho \text{ its value,}$$

$$fx = -\frac{\text{E}\dfrac{d^2y}{dx^2}}{\left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}};$$

which agrees with equation (1), Art. 91, for the common elastic curve.

Hence the curves into which the circular rod can be bent are the same as those which may occur in the case of the straight lamina.

If we take $DE = \frac{\text{E}}{fr}$, we shall have $EM = x$, and hence if EC, perpendicular to DE, meet the curve, it will cut it in a point of contrary flexure C.

98. PROB. *Fig.* 131. *To find what must be the natural form of a lamina aB, that a force F, acting perpendicularly at its extremity, may deflect it into a straight line AB.*

For the same reason as before r must be negative. Also ρ is infinite. And if a point p be, by the action of the force, brought to P, we have $AP = ap = s$, suppose; hence,

$$fs = \frac{\text{E}}{r}, \text{ or } rs = \frac{\text{E}}{f}; \text{ or, making } \frac{2\,\text{E}}{f} = a^2,$$

$$rs = \frac{a^2}{2};$$

which equation contains the property of the curve.

If pn be perpendicular on an, and if we make $an = x$, $np = y$, and the angle $ptn = \phi$, we shall have

$$\frac{dx}{ds} = \cos.\,\phi, \quad \frac{dy}{ds} = \sin.\,\phi, \quad r = \frac{ds}{d\phi}.$$

Hence $\dfrac{d\phi}{ds} = \dfrac{1}{r} = \dfrac{2s}{a^2}$; $\phi = \dfrac{s^2}{a^2}$, the arbitrary constant being $= 0$ if an be a tangent at a.

$$\therefore \frac{dx}{ds} = \cos.\,\frac{s^2}{a^2}, \quad \frac{dy}{ds} = \sin.\,\frac{s^2}{a^2};$$

and by integrating these expressions, we should have the values of x and y in terms of s.

We may integrate by expanding $\cos.\frac{s^2}{a^2}$ and $\sin.\frac{s^2}{a^2}$, and thus we obtain

$$x = s - \frac{s^5}{1.2.5a^4} + \frac{s^9}{1.2.3.4.9a^8} - \&c.$$

$$y = \frac{s^3}{1.3a^2} - \frac{s^7}{1.2.3.7a^6} + \frac{s^{11}}{1.2.3.4.5.11.a^{10}} - \&c.$$

which converge rapidly, except when s is very large in comparison with a.

Since the curvature increases in proportion to the distance from a, it is manifest that the curve will be a kind of spiral, which will tend to a point C with an infinite number of revolutions. The co-ordinates of this point C would be found, if we could find the values of x and y when s is infinite, which cannot be obtained from the series given above.

It makes no difference what point of the spiral we take for the point B. If we suppose that point and its tangent to be fixed, the portion of the curve Ba may always be bent into a straight line.

4. *Elasticity of Torsion.*

99. When a slender thread of metal, &c. is twisted, it tends to resume its natural condition, and would communicate angular motion to any body to which it is annexed, for instance, to a straight rod or rigid line fastened across it at right angles. A force acting on this rod may resist this tendency to motion, and produce equilibrium. The force necessary for this purpose is, as has been already mentioned, proportional to the angle through which the thread is twisted. Let there be a thread, perpendicular at C, fig. 132, to the plane of the paper. Let its upper extremity be fixed, and let Bb be a bar suspended at its lower extremity in a horizontal position. If this needle be turned out of the position Bb in which it would naturally hang, into any other Pp, the force which, acting at P in a horizontal plane and per-

pendicular to CP, would retain it in this position, will be as the arc BP, or as the angle BCP. If BC vary, the equilibrium will be preserved so long as the product of the force ($=F$) and distance BC remains the same; hence

$$F.BC \propto BCP.$$

If we call the angle BCP, θ, and the distance $CB=CP$, a, we shall have $Fa \propto \theta$, and $Fa=\epsilon\theta$, by properly assuming ϵ. The quantity ϵ is manifestly the value of the force F when the arm $BC=1$, and the angle $\theta=1$; it is different for different substances and masses, and may be considered as measuring the *elasticity of torsion*.

Problems in which elasticity of torsion enters present few difficulties; especially as there is no change of figure in the bodies which are concerned. We shall therefore only give one instance of their solution.

100. PROB. *Fig.* 132. *The extremity P of the bar whose natural position is Bb, is acted on by a repulsive force which varies inversely as the square of the distance from the center of force A, and is kept in its place by torsion; given its position, to find the force at A.*

Let the force of repulsion exerted by A be $\frac{f}{x^2}$; x being the distance AP. This force acts in the direction AP. Let it be resolved into two, one in the direction MP, of the lever CP, and the other in TP, perpendicular to CP. The former of these produces no effect to turn the lever CP, and the latter only is balanced by the torsion.

$$\text{Let } ACP=\theta, \text{ and } APT=ApP=\tfrac{1}{2}ACP=\tfrac{1}{2}\theta.$$

Hence the force which balances the torsion is $\frac{f}{x^2}\cos.\frac{1}{2}\theta$.

Let $CA=a$, and we have manifestly $x=AP=2a\sin.\frac{1}{2}\theta$. Hence the force which balances the torsion is $\dfrac{f\cos.\frac{1}{2}\theta}{4a^2\sin.^2\frac{1}{2}\theta}$.

Let now $ACB = \beta$, and the angle BCP, to which the torsion is proportional, will be $\theta + \beta$. The force of torsion will be $\epsilon(\theta + \beta)$ acting at P, perpendicular to CP; as is stated in last Article. Hence

$$\frac{f \cos. \frac{1}{2}\theta}{4a^2 \sin.^2 \frac{1}{2}\theta} = \epsilon(\theta + \beta);$$

whence

$$f = 4a^2\epsilon(\theta + \beta) \sin. \tfrac{1}{2}\theta \tan. \tfrac{1}{2}\theta.$$

If θ correspond to another position of Pp, f' being the force which retains it there, we have

$$f' = 4a^2\epsilon(\theta' + \beta) \sin. \tfrac{1}{2}\theta' \tan. \tfrac{1}{2}\theta',$$

whence

$$\frac{f}{f'} = \frac{(\theta + \beta) \sin. \frac{1}{2}\theta \tan. \frac{1}{2}\theta}{(\theta' + \beta) \sin. \frac{1}{2}\theta' \tan. \frac{1}{2}\theta'}.$$

If the arcs θ, θ' be very small, we may put the arc for its sine and tangent; and hence

$$\frac{f}{f'} = \frac{(\theta + \beta)\theta^2}{(\theta' + \beta)\theta'^2}.$$

If the points B and A coincide, $\beta = 0$,

$$\frac{f}{f'} = \frac{\theta \sin. \frac{1}{2}\theta \tan. \frac{1}{2}\theta}{\theta' \sin. \frac{1}{2}\theta' \tan. \frac{1}{2}\theta'};$$

and when θ, θ' are small,

$$\frac{f}{f'} = \frac{\theta^3}{\theta'^3}.$$

The combination supposed in this proposition agrees with the *Torsion Balance* of Coulomb, which has been employed for the purpose of measuring very small repulsive and attractive forces. In some cases the instrument was constructed with so much delicacy, that each degree of torsion required a force of only $\frac{1}{122400}$ of a grain.

CHAP. VII.

ON THE STRENGTH OF MATERIALS.

101. When solid bodies are made to undergo flexure beyond a certain degree, they break or undergo fracture: we now proceed to consider the force which is requisite, in order to produce this effect.

When bodies break, the fracture may *begin* either by the *tearing* of the extended parts, or by the *crushing* of the compressed parts. Thus if a beam be supported on two props, and broken by a weight hung to its middle point, the first destruction of the original texture of the beam may be either a *crack* on the under side of the beam, or a piece crushed or started out, on the opposite side.

As soon as the fracture is begun, it will have a tendency to extend across the beam; for the flexure will become greater and the power of resisting less by the failing of one part; therefore the remainder of the section of the beam will give way either by tearing or crushing.

We assume that the line which separates the parts torn and the part crushed, in a fractured section, will be the line which separated the parts extended and compressed, the instant before fracture.

102. In treating of the elasticity of materials, (Chap. VI.) it has been supposed that the resistance of the material to extension and to compression are on the same scale or modulus. In Art. 84, in the formula $t = E\dfrac{Rr}{VR}$, t represents the force

either of extension or compression, and E is the same in the two cases. But in reality the resistance of most materials to extension and to compression is different; whether we consider the resistance to flexure or to fracture.

Moreover it has been supposed that the resistance to extension and to compression in each fibre is proportional to the extension and to the compression; and the truth of this supposition can only be known by reference to experiment.

The former supposition, of the equality of the modulus of extension and compression, requires to be corrected, in order that we may apply our conclusions to practice with common materials, as wood and cast iron. It appears, for such substances, that in the case of moderate flexures the modulus of compression is generally less than the modulus of extension; and for the forces which are called into play when beams are broken, it appears that the difference of these moduli is still more considerable: as will appear in the subsequent articles.

The assumption that the forces of extension and compression vary each respectively as the extension and compression, appears to be more nearly true. For moderate flexures it was proved to be true by Mr Hodgkinson (*Manchester Memoirs*, Vol. IV. new series). He bent beams in such manners that in some cases they were capable of extension only, and in others of compression only; and he proved that in both cases, the deflexion produced was as the force applied transversely; whence it followed (Art. 86.) that the forces are as the extension, and as the compression respectively.

In the case of flexures produced in breaking, it follows, from Mr Barlow's experiments, that the same law is very nearly true, as will be shewn in subsequent articles.

103. PROP. *The modulus of elasticity being different for compression and extension, to find the position of the neutral line in any beam exposed to a transverse strain.*

Let the beam be fixed perpendicularly in a wall, and bent by a force F, acting at an arm l, in a direction parallel to the transverse section. The sum of the forces of compression and the sum of the forces of extension, which act on the two sides of the neutral line, must be equal to each other; for the transverse force which bends the beam, is parallel to the transverse section, and cannot balance any portion of either of these forces.

Let δa be any portion of the area of the extended transverse section, and x the distance of this portion from the neutral line, then the extension will be as x. Also let E be the force of extension of a fibre at a distance h from the neutral line, and let $\phi(x)$ be the function of the extension to which the force is proportional. Then the force exerted by the area δa will be

$$\frac{E\phi(x)}{\phi(h)}\delta a,$$

and the sum of all these forces is the whole force of extension.

In like manner if C be the force of compression of a fibre at the distance h from the neutral line, we shall have a similar expression for the force of compression at any point. Hence equating these expressions:

$$E \times \text{sum of all the } \phi(x) . \delta a = C \times \text{sum of all the } \phi(x) . \delta a.$$

The sum on the first side being taken for the extended, and on the second for the compressed area.

Cor. 1. If the force of the fibres be the same for all degrees of extension and compression; (Galileo's hypothesis;)

$$\phi(x) = 1,$$

$$E \times \text{extended area} = C \times \text{compressed area}.$$

COR. 2. If the forces of extension and compression be proportional to the extension and compression

$$E \times \text{sum of all the } x \,.\, \delta a = C \times \text{sum of all the } x \,.\, \delta a.$$

Therefore by the property of the center of gravity

$$E \times \text{extended area} \times e = C \times \text{compressed area} \times c:$$

e and c being the distances of the centers of gravity of the extended and compressed areas respectively from the neutral line.

Hence in this case

$$\frac{\text{extended area}}{\text{compressed area}} = \frac{Cc}{Ee}.$$

COR. 3. In a rectangular beam bent transversely, the center of gravity of each area is in the middle of its length. Hence, on the supposition of Cor. 2,

$$\frac{2e}{2c} = \frac{Cc}{Ee} \text{ and } \frac{C}{E} = \frac{e^2}{c^2} \text{ and } C : E :: e^2 : c^2.$$

In moderate strains the forces of extension and compression are nearly as the extension and compression. Hence this corollary is here applicable.

104. It appeared by Mr Hodgkinson's experiments, that in rectangular beams of fir, exposed to moderate flexure, the depths of the section extended and compressed were in the ratio of 169 to 190 nearly. Hence

$$E : C :: (190)^2 : (169)^2 :: 100 : 79 :: 5 : 4 \text{ nearly}.$$

In Mr Barlow's experiments fir beams were broken; and it then appeared that the areas extended and compressed, were as 3 to 5 nearly. Hence in this case

$$E : C :: 25 : 9 :: 11 : 4 \text{ nearly};$$

if the forces be in this case as the extensions and compressions, which it will hereafter appear they are, nearly.

The ratios from these different experiments are very different. It is possible, that in the very act of breaking a considerable change takes place in the proportion of the extended and compressed areas.

105. PROP. *The same suppositions being made as in the last Proposition, to find the position of the neutral line, in a beam, the section of which is an isosceles triangle, and which is bent in a plane perpendicular to the base of the triangle; the vertex of the triangle being in the extended side of the beam.*

We shall assume $\phi(x) = x$ as in Cor. 2, of last Proposition.

Resume the equation of last Proposition,

$$E \times \text{sum of all the } \phi(x) \,.\, \delta a = C \times \text{sum of all the } \phi(x) \,.\, \delta a.$$

If b be the base of the triangle, g and h the height of the compressed and extended portions respectively from the neutral line; we have, for the extended surface,

$$\delta a = \frac{b}{g+h}(h - x)\,\delta x, \text{ and since } \phi(x) = x,$$

The sum of all the $\phi(x) \,.\, \delta a$

$$\text{is } \frac{b}{g+h}\left(\tfrac{1}{2} hx^2 - \tfrac{1}{3} x^3\right) \text{ from } x = 0, \text{ to } x = h;$$

$$\text{that is, it is } \frac{b}{g+h} \cdot \frac{h^3}{6}.$$

For the compressed surface,

$$\delta a = \frac{b}{g+h}(h + x)\,\delta x,$$

and the sum of all the $\phi(x) \,.\, \delta a$ is

$$\frac{b}{g+h}\left(\tfrac{1}{2} hx^2 + \tfrac{1}{3} x^3\right), \text{ from } x = 0 \text{ to } x = g;$$

$$\text{that is, it is } \frac{b}{g+h}\left(\tfrac{1}{2} hg^2 + \tfrac{1}{3} g^3\right);$$

Hence

$$Eh^3 = C(3hg^2 + 2g^3); \quad \frac{E}{C} = 3\frac{g^2}{h^2} + 2\frac{g^3}{h^3}.$$

Hence $\frac{E}{C}$ being known, $\frac{g}{h}$ may be found.

COR. 1. The determination of the ratio $\frac{g}{h}$ from the above equation would require the solution of a cubic equation. The relation of $\frac{g}{h}$ and $\frac{E}{C}$ may be determined more simply, by means of the following Table:

$h = 12$	$C = 1728$	$C = 1$
$g = 12$	$E = 8640$	$E = 5.00$
11	7018	4.06
10	5600	3.24
9	4374	2.53
8	3328	1.92
7	2450	1.42
6	1728	1.00
5	1150	.66
4	704	.40
3	378	.22
2	160	.09
1	38	.02

It appeared in an experiment of Mr Barlow on an equilateral prism of fir, (*On the Strength of Timber*, 3d ed. p. 172) that g was .75 of an inch, and h was .982 of an inch.

Hence in this case $\frac{g}{h}$ was $\frac{9.17}{12}$, and by the Table, $\frac{E}{C}$ was a little greater than 2.5; and, as appears by interpolation, was 2.64 nearly.

106. Prop. *The same suppositions being made as in the last proposition, and the edge being on the compressed side, to find the position of the neutral line.*

The reasoning will be nearly the same as before. If h' and g' be the height of the compressed and extended portions from the neutral line, and E, C the force of extension and compression at the distance h' from that line,

$$E\,(3h'g'^2 + 2g'^3) = Ch'^3\,;$$

$$\frac{E}{C} = 3\frac{g'^2}{h'^2} + 2\frac{g'^3}{h'^3}.$$

Cor. 1. Hence the same Table as before, (Cor. 1, of last Prop.) will give the relation between $\frac{h'}{g'}$ and $\frac{C}{E}$, putting in the table

$$\frac{g'}{h'} \text{ for } \frac{g}{h} \text{ and } \frac{C}{E} \text{ for } \frac{E}{C}.$$

In an experiment of Mr Barlow on an equilateral prism of fir it appeared (p. 173) that g' was .39 of an inch, and h' was 1.342 inch. Hence in this case $\frac{h'}{g'}$ was $\frac{3.5}{12}$, and by the Table, $\frac{C}{E}$ was .31 nearly. This gives $\frac{E}{C} = 3.22$, which is not much different from the value obtained in the last Article, namely, 2.64.

If we had $g : h :: 5 : 6$ when the edge is extended, and $g' : h' :: .7 : 24$ when the edge is compressed, we should obtain from both cases nearly the same ratio of the force of extension to that of compression, namely, $E : C :: 3.2 : 1$ nearly.

The approximate coincidence shews that the supposition on which the above investigations proceed, namely, that the forces of extension and compression in each fibre are as the extension and compression, is nearly true up to the limit of fracture; supposing the neutral line, as shewn by the fracture, to be the neutral line before the fracture.

107. PROP. *Having given the position of the neutral line, and the absolute force of direct cohesion of the material, to find the force which applied transversely to any beam will break it.*

Let F be the force, and l the arm at which it is applied perpendicularly. The force F and the forces of extension and compression must balance each other about the neutral axis. Hence their moments must be equal, and we have, retaining the notation of Article 103, and using the abbreviation Σ to express "the sum of all the"

$$Fl = \Sigma E \frac{\phi(x)}{\phi(h)} x\,\delta a + \Sigma C \frac{\phi(x)}{\phi(h)} x\,\delta a.$$

Whence

$$Fl = \frac{E}{\phi(h)} \Sigma \phi(x) \,.\, a\delta a + \frac{C}{\phi(h)} \Sigma \phi(x) \,.\, x\delta a \ldots (2),$$

which is to be combined with the equation of Article 103;

$$E\,\Sigma\phi(x)\,\delta a = C\Sigma\phi(x)\,\delta a \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In general we may eliminate C by (1), and thence find the relation of E and F by (2).

In the case of fracture, $E\delta a$ is the force which will break a fibre, having a section δa, at the distance h from the neutral line. Therefore E is the force of direct cohesion for a surface 1, and is therefore known by proper experiments.

108. PROP. *Having given the position of the neutral axis in a rectangular beam, and the force necessary to break it; to find the law of the force of extension.*

Let the beam, fixed in a wall, be broken by the tearing of the extended surface, by means of a force F acting transversely at the extremity of the beam, the length of the beam being l. This force must, at the moment of fracture, balance the forces of extension and compression. Hence

$$Fl = \frac{E}{\phi(h)} \Sigma \phi(x) \,.\, x \delta a + \frac{C}{\phi(h)} \Sigma \phi(x) \,.\, x \delta a.$$

And in this case, h is the distance from the neutral line to the point of greatest extension, at which fracture begins, and E is the greatest force of extension which the substance can exert; or that with which it just yields to direct divulsion.

If we suppose $\phi(x)$ to be x^m, and the beam to be rectangular, the equation of the former proposition, Article 103, namely,

$$E \Sigma \phi(x) \delta a = C \Sigma (\phi x) \delta a,$$

will give us, using integration to find the sums,

$$E h^{m+1} = C g^{m+1};$$

g and h being the whole length of the sections of compression and extension measured from the neutral line.

Also on the same supposition, b being the breadth of the section, the equation (2) of last Article gives

$$Fl = E \times \frac{b h^{m+2}}{(m+2) h^m} + C \times \frac{b g^{m+2}}{(m+2) h^m};$$

whence, by the previous equation, we find

$$Fl = \frac{E b h^{m+2} + E b h^{m+1} g}{(m+2) h^m} = \frac{E b h}{m+2} (h+g);$$

$$m + 2 = \frac{E b h (h+g)}{Fl}.$$

Ex. It appeared by Mr Barlow's experiments (p. 168) that a fir beam 24 inches long and 2 inches square, fixed at one end in a wall, required a weight of 558 lbs. at its extremity to produce fracture. The neutral point or axis was at about $\frac{3}{8}$ the depth of the beam. The force of direct cohesion on a square inch of the same wood was 13000 lbs.

Therefore in inches $b = 2$, $h + g = 2$, $h = \frac{3}{8}$ of $2 = \frac{3}{4}$, $l = 24$; also $F = 558$ and $E = 13000$: hence

$$m + 2 = \frac{13000 \times \frac{3}{4} \times 2}{558 \times 24} = 3 \times \frac{13000}{13392}.$$

This is very nearly 3; therefore $m + 2 = 3$ and $m = 1$ nearly.

Thus the assumption of the previous Articles that the force of extension is as the extension is confirmed.

109. Prop. *The same suppositions being made as in the previous propositions, to find the force which, acting transversely as in Art. 108, will break a beam the section of which is an isosceles triangle; the edge being extended.*

Retaining the notation of the preceding Articles 105 and 106, we have equation (2) as before.

$$Fl = \frac{E}{h}\Sigma x^3 \delta a + \frac{C}{h}\Sigma x^2 \delta a.$$

Now for the extended area,

$$\delta a = \frac{b}{g+h}(h - x)\delta x.$$

Hence

$$\Sigma x^2 \delta a = \frac{b}{g+h}\left\{\frac{hx^3}{3} - \frac{x^4}{4}\right\} = \frac{b}{g+h}\cdot\frac{h^4}{12}$$

And for the compressed area,

$$\delta a = \frac{b}{g+h}(h + x)\delta x.$$

Hence

$$\Sigma x^2 \delta a = \frac{b}{g+h}\left\{\frac{hx^3}{3} + \frac{x^4}{4}\right\} = \frac{b}{g+h} \cdot \frac{4hg^3 + 3g^4}{12}.$$

Whence equation (2) becomes

$$Fl = \frac{b}{12h(g+h)}\left\{Eh^4 + C(4hg^3 + 3g^4)\right\}.$$

But by Art. 105,

$$C = \frac{Eh^3}{3hg^2 + 2g^3}.$$

Hence

$$Fl = \frac{Ebh^2}{12(g+h)}\left\{h + \frac{4h+3g}{3h+2g}g\right\}$$

$$= \frac{Ebh^2}{12(g+h)} \; \frac{3(g+h)^2}{3h+2g} = \frac{Ebh^2(g+h)}{4(3h+2g)},$$

which gives the value of F for the case of fracture.

Cor. 1. In nearly the same manner we shall find for the case in which the edge is compressed, the beam being broken by extension,

$$F'l = \frac{Ebg'^2(g'+h')}{4h'}.$$

Cor. 2. Hence the strength when the edge is extended is to the strength when the edge is compressed,

$$F : F' :: \frac{h^2(g+h)}{3h+2g} : \frac{g'^2(g'+h')}{h'}.$$

Let $g + h = g' + h' = c$, the depth of the beam. Therefore $3h + 2g = 2c + h$.

$$F : F' :: \frac{h^2}{2c+h} : \frac{(c-h')^2}{h'}.$$

In the cases referred to in Articles 105 and 106, it appeared that when $E : C :: 3.2 : 1$ nearly,

$$h \text{ was nearly } \frac{5}{11}c \text{ and } h' \text{ nearly } \frac{24}{31}c.$$

Hence

$$F : F' :: \frac{25}{27 \times 11} : \frac{49}{31 \times 24} :: 18600 : 14553 :: 372 : 291.$$

The forces which produced fracture in these two cases were found by experiment to be 370 and 313 pounds respectively (*Barlow*, pp. 172, 173).

When fracture is produced by compression we may in like manner determine the force of a beam, knowing the force which is to resist a given surface of the material acting directly.

110. When a uniform beam rests on two props and is pressed by a weight in the middle, the effect is the same as if it were fixed at the middle and acted on by a transverse force at its extremity equal to the pressure on each of the props, that is, to half the weight in the middle.

Prop. *When a weight is supported on any point of a beam resting horizontally on two props, the requisite strength of the beam at each point is as the rectangle of the segments of the length of the beam.*

Let a beam rest on two props, the length between the props being l: and let a weight W rest on a point of the beam, the distances of which point from the props are p and q. The pressures on the two props are respectively

$$\frac{Wp}{l} \text{ and } \frac{Wq}{l};$$

and the moment of each of these to turn the corresponding end of the beam round the point where W rests is

$$\frac{Wpq}{l}.$$

Hence the strength of the beam at different points must be as pq, the rectangle of the segments of the beam.

INTRODUCTION

TO

DYNAMICS.

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INTRODUCTION

TO

DYNAMICS,

CONTAINING

THE LAWS OF MOTION

AND

THE FIRST THREE SECTIONS OF THE PRINCIPIA.

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CONTENTS.

ERRATA.

Page 3, line 18, *dele* finite.
—— 56, — 18, *after* drawn, *insert* parallel to $C E$.
—— 57, — 12, *for* body *read* force.
—— — — 23, *after* whence *insert* ultimately.
—— 58, — 5, *for* by *read* in.
—— 61, — 12, *for* V^2 *read* V *twice.*
—— 62, — 13, *for* $4 C . D^2$ *read* $4 C D^2$.
—— 63, — 2, *from bottom*, *for* center C *read* new centre C'.

PREFACE.

FINDING myself called upon for a new Edition of the Treatise on Dynamics, I have judged it advisable to incorporate with it a considerable portion of the Principia of Newton: and being hence led to connect with my plan the first three Sections of the Principia, I thought that I should best consult the convenience of the reader, by publishing that part of the work separately.

Besides the value of these sections as a portion of Dynamics, they have, I conceive, other very great claims to attention as a part of our course of mathematical study. The first section of the Principia is eminently instructive with reference to the fundamental principles of the Differential Calculus. At the point where the student enters upon that province of mathematics, he has, brought under his notice, (if he be one of those who look for sound logic in the science,) a set of views and of modes of reasoning which have not occurred in the earlier parts of his progress. The passage from definite to indefinite magnitude, from discrete to continuous change, is to be made, and the consequences of it are in some way to be traced. This may be done in a number of different paths, according to the side on which the reasoner advances to the point: but on all the roads nearly the same difficulties, or at least transitions, occur, in one stage or other; and these steps ought to be steadily looked at and clearly mastered. I do not think this can anyhow be done better or more

effectually than by a study of the Lemmas of the first section. Newton himself claims for it the praise of avoiding at the same time the tedious length of the indirect demonstrations of the ancients, and the harsh hypotheses and unsatisfactory logic of indivisibles, and, as he might have added, if he had written later, of infinitesimals. And such praise does not appear exaggerated: for while the logic of the method of ultimate ratios may, by a little care, be made unobjectionable, propositions are, by means of it, proved and applied with great compendiousness and facility. It is as generally applicable, and as free from superfluous steps, as any synthetical method can be expected to be. And it is very desirable that the mathematical student, before he rushes forwards to differentiate and integrate upon the slightest provocation, should employ some thought in understanding the construction and trustworthiness of the instrument which he is so familiarly to use.

When in our mathematical career we arrive at the consideration of the properties of curves in general, and of the operation of variable forces, the doctrine of ultimate ratios, or some equivalent method, is summoned to our aid, to effect a *denouement* of difficulties which more elementary modes of calculation are inadequate to solve: and such are the purposes to which this doctrine is applied in the Principia. In order to make the reasoning of this application systematic, I have introduced definitions of those notions which do not occur in the more elementary parts of geometry and mechanics, such as continued curvature, finite curvature, velocity, force. It appeared to me that I could no otherwise make these definitions strict, and the deductions from them logical, than by founding the definitions themselves upon the doctrine of ultimate ratios: I have done this, endeavouring to give them the simplest forms of which their purpose admits.

I have also stated the fifth Lemma as a postulate, a character which I think it may not unfitly occupy: at any rate it cannot be proved from the principles of this first section, without a most glaring example of a vicious circle of reasoning. If we examine our conception of *similarity* in figures, it will perhaps be found to be almost as distinct and simple as our conception of *equality*. If the reader is persuaded that this is so, the postulate will readily be conceded. If he is of a different opinion, the discussion is of an abstract and metaphysical kind, not properly belonging to my present purpose.

Newton's proofs of his Lemmas are in many cases very briefly exprest. When we attempt to develope his reasoning we should be careful not to insert any steps which will not bear examination. Thus in proving the fourth Lemma, if we suppose two ratios $a : p$ and $b : q$ to be each *ultimately* as $m : n$, we cannot properly deduce the relation $a + b : p + q :: m : n$, *ultimately*, by applying the propositions *componendo*, *alternando*, &c. which have only been proved for ratios *actually* the same, and not for those *ultimately* the same.

In proving the tenth Lemma, the definitions of velocity and force were necessarily introduced: and in the succeeding sections, the second Law of motion. I have therefore placed in this publication the exposition of the fundamental principles and laws of Dynamics. In doing this I have stated *three* laws of motion, corresponding to Newton's three Laws: and in this instance I have not acted from a wish to accommodate my course of dynamical deduction to that of the Principia, but from a persuasion that this mode of presenting the principles of the subject is the right one. In the works of many authors, and especially in those of the great mathematicians of France, to whom this science owes so much, it is asserted that there are but *two* fundamental laws of motion, borrowed from experience; namely, the law of inertia, and the principle that force

is as velocity. I have explained elsewhere my conviction that the simplification which thus attempts to reduce the two latter Laws of motion to one, is not consistent with the mode in which these laws are established. I have, in the following pages, briefly touched upon this discussion, and have there given what I believe to be a just statement of the point at issue. "The *second* Law of motion is proved by the experiment suggested by Laplace; namely, by the fact that the oscillations of the same pendulum are *equally quick in all azimuths*. The third Law is proved by experiments which shew that the *time* of oscillation is proportional to the *square root of the length* of the pendulum. One of these laws cannot be deduced from the other, except one of these facts can be shewn to be implied in the other."

I am well aware that such discussions cannot be expected to excite much interest. It is a peculiar feature in the fortune of principles of such high elementary generality and simplicity as characterise the laws of motion, that when they are once firmly established, or supposed to be so, men turn with weariness and impatience from all questionings of the grounds and nature of their authority. We often feel disposed to believe that truths so clear and comprehensive are necessary conditions, rather than empirical attributes, of their subjects: that they are legible by their own axiomatic light, like the first truths of geometry, rather than discovered by the blind gropings of experience. And even when the experimental foundation of these principles is allowed, there is still no curiosity about the details of the induction by which they are established. The process of *deduction*, of reasoning downwards from principles, fills the mind at every step with a confidence in its own workings, a consciousness of certainty, a distinctness of perception, a feeling of superiority to all more vague and doubtful impressions, which give a

peculiar charm to this employment, and have often tempted men to pursue it when the truth obtained was of no value; and even to apply it when it could not possibly lead to truth. But the process of *induction*, by which we arrive at principles, is one which, though at least as important to the progress of science, possesses no such fascination. To scrutinize this process after it has been successfully executed, we must put the mind in an attitude of doubt on doctrines not only certain, but the foundation of long trains of certainty; we must try to conceive ourselves ignorant of that which we most familiarly assume; in short we must attempt to guess a riddle of which we already know the answer. It is not surprising therefore that this should not be a favourite occupation with speculative men; and I shall be little disappointed if most readers are content with knowing that all mathematicians have now given their assent to the same general doctrine concerning the motion of bodies, and if they feel indifferent whether this doctrine be subdivided into two laws or three.

Occasionally, however, I may have a reader whom such speculations interest; and in defence of our common taste, I may plead, that the simplicity and generality of mechanical laws is no infallible proof of their truth, even when their authority has long been consecrated by the assent of large bodies of readers. The Aristotelian might reasonably claim some of the beauty of simplicity for his laws of motion, when he maintained that violent motion had a constant tendency to decrease, while natural motion could go on undiminished; or when he held that heavier bodies fell more quickly than light ones, because their tendency to descend, as shewn in their weight, was greater: and for the votaries of studies where one of our great boasts is that we are certain, and know the grounds of our certainty, it seems to be no inap-

propriate inquiry, how, when the Aristotelian was as positive as we can be, we satisfy ourselves that he was wrong and we are right.

But, whether or no the reader may attach any importance to the proof of the Laws of motion, I hope that, starting from these principles, he will find their application in the propositions which compose Newton's second and third sections, clearly presented. I have endeavoured to do this, adhering very closely in most cases to Newton's synthetical proofs. There appear to me to be very good reasons for making these, so far as the second and third sections go, a part of the reading of the young mathematician. Some ground for this may be found even in their historical importance. No student of the mathematical doctrine of the motions of the heavens, at least no English student, ought to be ignorant of the manner in which its leading propositions were established by the great discoverer of the true laws of the universe. But, independently of this claim on our notice, the geometrical treatment of these propositions is likely to be of service to the learner. He will generally conceive objects and their connexions far more distinctly when they are presented to him in a geometrical form, and when he reasons by means of representations and notions which are resemblances of the things treated of, than when these objects are replaced by arbitrary symbols, and when the rules of combination of these symbols exclude from his thought the relations on which his inferences really depend. Though, therefore, the analytical mode of treating Dynamics is the only method which can now answer the requisitions of natural philosophy, it may be well that the student should begin by a few geometrical propositions on the subject; and thus be prepared to dissipate the usual obscurities arising from confused conception.

Moreover the great importance of some of the propositions contained in these sections renders it desirable, for the sake of those who may acquire a little mathematics only, that these truths should be established with as little of scaffolding and apparatus as may be. The Differential Calculus is not necessary for a person who would merely understand the mechanics of the skies, though that instrument is indispensable for one who would himself examine or extend the calculated results. In another volume I hope to make it appear that Newton's modes of reasoning on the problem of three bodies are well fitted to the purposes of those who wish to study the leading phenomena of the lunar perturbations, and who nevertheless shrink from the laborious process of the analytical solution; in which we have to expand our series, to collect and scrutinize our terms, to substitute and calculate, with no relief from any intermediate perception of the causes and nature of the motions which we are thus tracing to their ultimate effect upon the place of the body.

But though I have thus given Newton's propositions in these three sections, I have not added the train of other propositions, concerning angular velocities, centrifugal forces, &c. which are easily deducible from them, and often associated with them in our studies in this place. There is little in the intrinsic value or utility of these investigations to justify my giving any space to them; and when there lie in the mathematician's road so many really important applications of his science, which may well employ all his skill, it would be to trifle very unprofitably with his time, if I were to recommend these useless speculations to his attention. In synthetical investigations the student may shew his proficiency either by presenting clearly the author's reasoning, or by himself employing the principles which he has learnt, for the solution of new problems. He may merely *understand* and

keep in mind the doctrines he has studied, or he may draw *inferences* from them; and both kinds of attainment are properly considered as evidences of merit. But to require him to *understand* and keep in mind the *inferences* made by other students; to make the *evidence* of the study of one generation the *subject* of the study of the next; is to introduce a system of endless and aimless trifling. This consideration appears to me to make all published or circulated collections of problems and deductions on subjects *synthetically* treated, something worse than useless, so far as our system of mathematical instruction is concerned. I have, therefore, confined myself almost exactly to the propositions and corollaries as they stand in the Principia, omitting a few, which are not wanted according to the manner in which I have treated the subject. The only exception to this rule (besides a few corollaries) is the proposition which I have added at the end of the second section. I have admitted it in consequence of its great geometrical elegance, and because it completes the theory of the ellipse about the centre, as the proposition at the end of the third section does the theory of the ellipse about the focus. If, however, any one wishes to apply the rule of exclusion without exception, he can easily, after this warning, pass over this proposition. In the 17th proposition, I have substituted, instead of Newton's reasoning, another method which appears to me equally geometrical and more simple.

Nor have I attempted to illustrate and explain Newton's words and expressions. To do so would again have led to a train of tedious and unprofitable disquisitions, of a nature more suited to the philological critic than to the mathematician. Such verbal illustration and analysis is indeed an employment which has its attractions, both for the writer and the reader; and accordingly the spirit of commentatorship has generally been sufficiently ready to fasten upon the objects of the

intellectual admiration of mankind. Its flowers and weeds seem to spring up luxuriantly about the wheels of the car of genius, the moment there is a pause in the career. But these accompaniments are more likely to obstruct than to further the progress of real knowledge. In physical inquiries at least, the mere commentator on the works of others is not likely to make his labours of much value. In the speculations of taste and criticism indeed, or in subjects where the rule of action or of opinion depends mainly upon authority, he may be well employed and in his place. He may point out connexions else unseen; he may present distinctly to our consciousness that which was obscurely felt; he may exhibit relations of order and subdivision which facilitate the application of our rules. The analogy between the studies of the philologer, or antiquary, or casuist, and the pursuits of physical science, is perhaps too vague and imperfect to be dwelt on: but, so far as we assume it to exist, we may say that in the former class of inquiries the thoughts which have passed through the minds of other men and the words they have used are, to a great extent, the *phenomena* with which we are concerned, and which are to be the basis of our knowledge. And the process by which the critic unfolds all the recondite connexions or emotions or convictions through which the sympathy or assent of men has been won, corresponds, not to our reasonings *from* assumed axioms, but to those in which we try to ascertain what our axioms ought to be. His disquisitions seem to be attempts to perform that higher function of reasoning,—the ascent to general principles,—the discovery of the grounds of truth,—which in physics is termed Induction. But if in such researches on such subjects, the analysis of one man's thoughts by the labour of another may be of use, the case is different in mathematical works, where, besides that our phenomena are solely

the objects of sense, the process of investigation is entirely one of Deduction. Here our facts are to be established by observation and experiment, not by thought or opinion: here when we have once obtained our principles and general laws, we reason downwards from them; and in doing this there can be no truth contained in the conclusion, which is not involved in the premises: good logic is the one thing requisite; and no name can convert bad logic into good, nor any authority add to the evidence of demonstrated truth: here therefore the commentator's labour must be wasted. In the former cases, where we are still to seek for the first principles of systematic knowledge, we are like men who conduct their research in the perplexing obscurity of a protracted night; and the critic's torch, according to its degree of light, is a welcome aid: but in subjects which are already brought to the clearness of scientific intuition, the commentator's cares would resemble an attempt to throw the light of some particular lamp on an object already lying obvious under the beams of the risen sun.

Nothing could be more injurious to the progress of good mathematics and sound natural philosophy among us than to foster the opinion that it is an ultimate object of our studies to illustrate any book, even the Principia; instead of considering the book one instrument amongst others in the study of nature. I hope shortly to be able to lay before the reader a new edition of the Treatise on Dynamics, in which the most valuable of the Propositions of the Principia are introduced as part of the general course of investigation: a plan which, upon the considerations above stated, appears to me the most likely to be useful to the student of applied mathematics.

TRINITY COLLEGE, *March* 17, 1832.

INTRODUCTION TO DYNAMICS.

Section I.

DEFINITIONS, PRINCIPLES, AND LEMMAS.

Subsection I. Geometrical *Definitions, Postulates, and Lemmas.*

[Newton. Principia. *Book I. Section I.*]

In the following reasonings, certain hypotheses are assumed, (as that two points are taken in a curve, near to each other, or that a finite magnitude is divided into many small parts,) and certain constructions are made upon these hypotheses. The hypothesis is then extended indefinitely, the spaces and numbers which it involves being supposed to become greater or smaller than any given magnitudes; (for instance the two points in the curve are supposed to approach indefinitely near to each other; or the parts of the finite magnitude are supposed to become indefinitely numerous and indefinitely small.) The properties of the construction above mentioned, will, in consequence of this extension of the hypothesis, approach constantly to certain properties, which are the properties in the *ultimate form of the hypothesis.*

The values of any of the magnitudes so deduced from a construction are called their *ultimate* or *limiting values;* and ratios so deduced are called *ultimate* or *limiting ratios.*

These are sometimes also called *prime* ratios, the hypothesis being supposed to be extended *from* its ultimate form, instead of *to* it.

The quantities of which we have to consider the ratios, may vanish in the ultimate form of the hypothesis. Their ratio is then sometimes called their *vanishing ratio.*

Objection 1. There are no ultimate values or ratios: for by an indefinite extension of the hypothesis we cannot arrive at definite properties.

Answer. By an indefinite extension of the hypothesis we do approach, in general, to definite properties, as will be seen in succeeding propositions. The following may be taken as a simple example. Let there be inscribed in a given circle regular polygons of 4, 8, 16, 32, 64, 128, 256, 512, 1024 &c. sides and so on, doubling the number indefinitely. We then approach perpetually to this definite proposition;—that the inscribed polygon is equal to the circle.

Objection 2. There is no vanishing ratio; for before the quantities vanish, it is not the ultimate ratio; and when they have vanished, they have no ratio.

Answer. The vanishing ratio is neither the ratio of the quantities before they vanish, nor after they have vanished, but the ratio in which they vanish. It is the ratio to which their ratio approaches perpetually and indefinitely, while the hypothesis approaches its ultimate form.

Objection 3. We cannot have the ultimate ratio of vanishing quantities. For if the ultimate ratio be given, the ultimate magnitudes are given; and therefore the quantities do not ultimately vanish. And hence quantity would consist of ultimate and indivisible portions, which is false.

Answer. The ultimate ratio is given, though the ultimate magnitudes be not given; for the ultimate ratio is not the ratio which the quantities have in an ultimate state, but the ratio to which their ratio tends, while the hypothesis tends to its ultimate form.

The word *ultimately*, introduced in propositions, implies that they are true in the ultimate form of the hypothesis.

Lemma I. (*A*) *Quantities which constantly tend towards equality while the hypothesis approaches its ultimate form, and of which the difference, in the course of this approach, becomes less than any given magnitude, are* ultimately *equal.*

If the values of the quantities in the ultimate form of the hypothesis be not equal, let them differ by a magnitude D. But in the course of the approach of the hypothesis to its ultimate form, the difference of the quantities becomes less than the given magnitude D, while they are constantly tending towards equality. Therefore the difference of the quantities in the ultimate form of the hypothesis cannot be D. Therefore *ultimately* they have no difference, and are equal.

(*B*) *If Ratios constantly tend towards equality while the hypothesis approaches its ultimate form, and if certain factors, in the course of this approach, are* ultimately *in a ratio of equality, the ratios themselves are ultimately equal.*

Let the ratio $A \,.\, X \,:\, B \,.\, Y$ constantly tend towards equality with the ratio $M : N$; and let the finite factors X, Y be *ultimately* equal; then *ultimately*, $A : B :: M : N$.

If this be not true, let $A : B :: M : N(1 + D)$ where D does not *ultimately* vanish. Therefore $A \,.\, X : B \,.\, Y :: M \,.\, X : N \,.\, (1 + D)\, Y$; and the ratio $M \,.\, X : N \,.\, (1 + D)\, Y$ constantly tends to equality with the ratio $M : N$; or the ratio $X : (1 + D)\, Y$ constantly tends to equality with the ratio $1 : 1$. Hence the ratio $X(1 + D) : Y(1 + D)$ or $X : Y$ constantly tends to equality with $1 + D : 1$. But *ultimately* X is in a ratio of equality to Y, which is impossible except D vanish. Therefore there is no magnitude D; and therefore *ultimately* $A : B :: M : N$.

Lemma II. *If in any curvilinear space, bounded by a curve and its two co-ordinates, be inscribed any number of adjacent parallelograms, having their sides parallel to the co-ordinates, and having their bases, along one of the co-ordinates, equal :—*

And if the breadths (or equal sides) of these parallelograms be diminished and their number increased indefinitely :—

The sum of the inscribed parallelograms, that of the parallelograms similarly circumscribed, and the curvilinear space itself will be ultimately *equal.*

Fig. 1. Let $AacE$ be the curvilinear space; Ab, Bc, Cd, the inscribed parallelograms; Al, Bm, Cn, Do, the parallelograms similarly circumscribed. The excess of the sum of the latter above the sum of the former, is the sum of the partial excesses, Kl, Lm, Mn, Do; and this is manifestly equal to Al, because the bases Lc, Md, DE are each equal to AB.

When the breadth AB is diminished indefinitely, and the number of the parallelograms indefinitely increased, the excess Al, of the circumscribed above the inscribed parallelograms, decreases; and the two sums tend constantly to equality. And in approaching the ultimate form of the hypothesis, the difference Al becomes less than any finite magnitude, because the side Aa remains constant and the side AB is indefinitely diminished. Therefore the conditions of Lemma I. (A) are exactly satisfied, and the two quantities, the sums of the inscribed and of the circumscribed parallelograms, are *ultimately* equal.

The curvilinear space is always greater than the sum of the inscribed and less than the sum of the circumscribed parallelograms. Therefore the difference between this space and either sum is always less than the difference of the two sums. Therefore the same conditions and reasoning apply to the curvilinear space, as to the two sums of parallelograms, and it is *ultimately* equal to either of the sums.

Lemma III. *If the same hypothesis be made, except that the breadths of the parallelograms are not supposed equal:—*

And if the breadths be all diminished indefinitely as before:—

The same proposition is true as in Lemma II.

Fig. 1. Let AF be the greatest of the breadths of the parallelograms; and let the parallelogram $FAaf$ be com-

pleted. If the parallelogram Af be divided by the lines bK, cL, dM, each of the partial excesses Kl, Lm, Mn, Do, will be either less than the corresponding portion of the parallelogram Af, or equal to it. Hence the whole excess will be less than the parallelogram Af.

When all the breadths are diminished indefinitely, the sums of the inscribed and circumscribed parallelograms, and the curvilinear space, tend to equality, as before. And the difference is always less than Af, and Af *ultimately* becomes less than any finite magnitude. Therefore *ultimately* the inscribed and circumscribed parallelograms and the curvilinear space are all equal.

Cor. 1. Every portion of the ultimate sum of the parallelograms is equal to the corresponding portion of the curvilinear space.

Cor. 2. The polygonal figure contained by the chords ab, bc, cd, dE, is greater than the inscribed and less than the circumscribed series of parallelograms; and hence this polygonal figure is ultimately equal, in all its parts, to the curvilinear figure.

Cor. 3. The same may be said of the polygonal figure formed by drawing tangents to the curve at the points a, b, c, d, E.

Lemma IV. *If in two figures be inscribed, as in Lemmas* II *and* III, *two series of parallelograms equal in number:—*

And if, when the breadths of these parallelograms are diminished and their number increased indefinitely, the ultimate *ratios of the parallelograms in the one figure to those of the other, each to each, be all the same:—*

The curvilinear figures are also in the same ratio.

Fig. 2. Let AaE, PpT be the two figures; and let the ratios $a : p$, $b : q$, $c : r$, $d : s$, &c. of the corresponding parallelograms be *ultimately* all the same.

Let this ratio be $m : n$. And let

$$\begin{aligned} a : p &:: m : n\ (1+\alpha) \\ b : q &:: m : n\ (1+\beta) \\ c : r &:: m : n\ (1+\gamma) \\ &\text{\&c.} \end{aligned}$$

where α, β, γ, &c. are quantities which vanish in the ultimate form of the hypothesis. Hence

$$\begin{aligned} a\ (1+\alpha) : p &:: m : n \\ b\ (1+\beta) : q &:: m : n \\ c\ (1+\gamma) : r &:: m : n \end{aligned}$$

Therefore

$$a\ (1+\alpha) + b\ (1+\beta) + c\ (1+\gamma) + \text{\&c.} : p+q+r+\text{\&c.} :: m : n.$$

Therefore $a + b + c + \text{\&c.} + a\alpha + b\beta + c\gamma + \text{\&c.} : p + q + r + \text{\&c.} :: m : n$, or $(a+b+c+\text{\&c.})\ (a+b+c+\text{\&c.}+a\alpha+\text{\&c.}) : (p+q+r+\text{\&c.})\ (a+b+c+\text{\&c.}) :: m : n$.

Let μ be the greatest of the quantities α, β, γ &c. Therefore $a\alpha + b\beta + c\gamma + \text{\&c.}$ is less than $a\mu + b\mu + c\mu + \text{\&c.}$ Therefore $a + b + c + a\alpha + b\beta + c\gamma + \text{\&c.}$ always differs from $a + b + c + \text{\&c.}$ less than $a + b + c + \text{\&c.} + a\mu + b\mu + c\mu + \text{\&c.}$ does; that is, less than $(a + b + c + \text{\&c.})\ (1 + \mu)$ does.

Therefore in approaching the ultimate state, the factors $a + b + c + \text{\&c.} + a\alpha + \text{\&c.}$ and $a + b + c + \text{\&c.}$ are more nearly in a ratio of equality than $(a + b + c + \text{\&c.})\ (1 + \mu)$ and $a + b + c + \text{\&c.}$: that is than $1 + \mu : 1$.

But in approaching the ultimate state μ vanishes. Therefore ultimately, by Lemma I. (*B*)

$$a+b+c+\text{\&c.} : p+q+r+\text{\&c.} :: m : n,$$

also $\quad \text{figure } A : a+b+c+\text{\&c.} :: 1 : 1$ *ultimately*
$\quad p+q+r+\text{\&c.} : \text{figure } P. :: 1 : 1$ *ultimately* $\Big\}$ Lemma II. and III.

Therefore *ultimately*, figure A : figure P :: $m : n$ Lemma I. (*B*.)

Cor. Hence if two quantities of any kind be divided into the same number of parts; and those parts, when their number is increased and their magnitude diminished indefinitely, have a given ratio each to each: namely, the first part of the one quantity to the first part of the other; the second to the second, and so on: these two quantities will be in the given ratio.

For the parallelograms in the Lemma being taken to represent the parts, the figures will represent the quantities in question, and their ratio will therefore be the ultimate ratio of the parts.

Lemma V. Postulate concerning similar Figures.

In similar figures, corresponding lines, whether straight lines or portions of the curves, are proportional: and the areas of corresponding parts are as the squares of homologous lines.

The properties of figures do not depend on their absolute magnitudes. Hence we may conceive the magnitude of a figure changed, the form remaining the same, and the properties will continue unaltered. Hence the ratio of any two lines in the figure will be the same after this change as before. But by this change we obtain figures similar to the original figures. Hence the corresponding lines in similar figures have the same ratios, or are proportional.

The rectilinear figures similarly described in similar curves will be similar polygons; and these are (Euc. vi. 20) in the ratio of the squares of homologous lines. The curvilinear figures are, by Cor. 2 of Lemma iii, and by Lemma iv, in the ultimate ratio of such polygons, and are therefore also as the squares of the homologous lines.

DEFINITION OF CONTINUED CURVATURE.

If a point be taken in a curve, and chords be drawn from this point to two other points, one on each side of it:—

And if the latter two points be supposed to approach indefinitely near to the first point:—

Then, if the two chords *ultimately* coincide in direction, the curve at that point has CONTINUED CURVATURE.

The *ultimate* position of either chord is the TANGENT to the curve at that point.

COR. 1. No line can be drawn through any point of the curve, between the tangent at that point and the curve.

COR. 2. Every line drawn through any point of the curve, on the same side of the tangent as that on which the curve is, must, in approaching the tangent, cut the curve in another point.

LEMMA VI. *The angle contained between the chord and the tangent*, ultimately *vanishes at a point of continued curvature.*

This appears from the above definition.

LEMMA VII. *If through any point in a curve of continued curvature, there be drawn a tangent, and also a chord meeting the curve in a second point: and if the tangent be limited by a* secant (*or line cutting the curve*) *through the second point:—*

And if the second point approach indefinitely near to the first:—

The ultimate ratio of the arc, the chord, and the tangent is a ratio of equality.

Fig. 3. Let AD, AB be the tangent and the chord of ACB, BD the secant. Let bd be parallel to BD, and let the arc acb be always similar to ACB: then Ad will be a tangent to Acb, because the homologous lines in similar figures must have similar relations of position, (Lemma v.)

Let Ad, Ab remain finite, while B approaches indefinitely to A. Then by Lemma VI, the angle bAd or BAD *ultimately* vanishes: and therefore *ultimately* the tangent Ad, the chord Ab, and the arc Acb coincide and are equal. But by Lemma v. these are always in the ratio of the tangent AD, the chord AB, and the arc ACB: which three are therefore *ultimately* equal.

COR. 1. Fig. 4. If through the second point B, be drawn BF parallel to the tangent, and AF a secant through A, BF will *ultimately* be equal to the arc ACB; for BF is equal to AD.

COR. 2. If several secants be drawn through B, as BD, BE, or through A, as AF, AG, all the *abscissas* (or lines cut off) AD, AE, BF, BG, will *ultimately* be equal to the arc, and therefore to each other.

COR. 3. Hence these lines, in the ultimate form of the hypothesis, (namely on the supposition that B approaches indefinitely near to A,) may be taken for each other.

N. B. The secants must all make with the tangent angles which are ultimately finite.

LEMMA VIII. *If two lines, drawn through the first and second points in the curve, make with the chord, the curve, and the tangent at the first point, three triangles:—*

And if the second point approach indefinitely near to the first:—

The three triangles are ultimately similar as to form, and in a ratio of equality.

Fig. 3. Let AR, BR be the two lines; and making the same construction as in last Lemma, let dbr be parallel to DBR.

Let Ad remain finite when B approaches indefinitely to A; therefore Ab, Acb, Ad coincide; and the three triangles rAb, rAd, $rAcb$, are *ultimately* identical as to form, and in a ratio of equality. Therefore the three triangles RAB, RAD, $RACB$, (which are always similar and proportional to the others by Lemma v.) are *ultimately* similar as to form, and in a ratio of equality.

COR. Hence these triangles, in the ultimate form of the hypothesis, and in all reasonings founded upon it, may be taken for each other.

LEMMA IX. *If through a fixed point in a curve there be drawn a secant, and if through two variable points of the curve there be drawn* ordinates *(that is, parallel lines)*

meeting the secant, and making two triangles (each having a curvilinear side:)—

And if these two variable points approach at the same time indefinitely near the fixed point:—

The two triangles will ultimately *be in the ratio of the squares of their corresponding sides.*

Fig. 5. Let A be the fixed point, B, C, the variable points, BD, CE the ordinates to the secant AE. Let c be taken in AC produced, and let the curve Abc be always similar to ABC, ABb being a straight line. Let also ordinates bd, ce be drawn meeting the secant; and let $AFGfg$ be a tangent to ABC, and therefore to Abc (Lemma v.)

Let Ae remain constant while C and B approach indefinitely to A. The angle cAg or CAG will vanish; the triangles Abd, Ace will *ultimately* coincide with the triangles Afd, Age; and the areas Abd, Ace will ultimately be in the same ratio as the areas Afd, Age; namely, by similar triangles, as the squares of Ad, Ae. But the areas ABD, ACE are proportional to the areas Abd, Ace; and AB, AD are proportional to Ab, Ad, by Lemma v. Hence it follows that ABD, ACE, are ultimately as the squares of AD, AE.

Lemma x, is given in the next Subsection.

DEFINITION OF FINITE CURVATURE.

If a circle be drawn touching a curve at any point, and cutting it in a second point:—

And if the second point approach indefinitely to the first:—

Then, if the diameter of the circle be *ultimately* finite, the curve has FINITE CURVATURE at the first point.

The *ultimate* form of the circle mentioned in the definition is called the CIRCLE OF CURVATURE. And the CURVATURE of the curve is the *same* as that of this circle.

Cor. 1. Fig. 6. Let A, B, be the two points, AG perpendicular to the tangent, BG perpendicular to the chord. The circle described on the diameter AG, touches the curve

at A, and meets it at B. Hence, by the Definition, AG and BG being thus drawn, AG is *ultimately* finite, if the curvature at A be finite.

Cor. 2. No circle can be drawn through the point of contact, so as to pass between the curve and its circle of curvature.

Lemma XI. *In curves of finite curvature, the subtense of the angle of contact is ultimately as the square of the conterminous arc.*

Case 1. Let the subtense be perpendicular to the tangent.

Fig. 6. Let AB be any arc, AD its tangent, BD the subtense of the angle of contact.

Let Ab be any other arc, bd its subtense: and let BG, bg be drawn perpendicular to the chords AB, Ab, meeting the normal AG in G, g; AI the ultimate value of the chord AG or Ag. It is easily shewn that $AG . BD = AB^2$, $Ag . bd = Ab^2$. Now $AG : Ag :: AG . AI : AI . Ag$: and the ratios $AG : AI$ and $AI : Ag$ being *ultimately* ratios of equality, (by the Definition) we have *ultimately* $AG : Ag$, a ratio of equality. Therefore $AG . BD : Ag . bd$ is *ultimately* $:: BD : bd$. Hence *ultimately*

$BD : bd :: AB^2 : Ab^2 ::$ arc AB^2 : arc Ab^2 by Lemma viii.

Case 2. Fig. 7. Let the subtense BD be inclined at a given angle to AD. The ratio of BD to bd will be the same as that of the perpendicular subtenses BC, bc; and therefore, by Case 1, the same as the ratio of AB^2 to Ab^2.

Case 3. Fig. 7. Let the subtense BD be inclined to AD at an angle, variable according to any law, but ultimately finite.

Let BC, bc be drawn parallel to the ultimate position of the subtense BD. Then BD is to BC *ultimately* in a ratio of equality. In the same manner bd is to bc *ultimately* in a ratio of equality. Therefore *ultimately*

$BD : bd :: BC : bc :: AB^2 : Ab^2$, by the last Case.

Cor. 1. Fig. 7. If AV be a chord of the circle of curvature parallel to the subtense BD, ultimately $AV . BD = AD^2$. For in the circle ABW, the angle BWA is always

equal to BAD, and BAW to ABD. Therefore the triangles ABD, WAB are similar:

therefore $AW : AB :: AB : BD$, and $AW.BD = AB^2$.

And ultimately AW becomes AV, AB becomes equal to AD, and the circle becomes the circle of curvature AOV:

Therefore $\quad AV.BD = AD^2$ ultimately.

Cor. 2. Fig. 9. If DA, DC are two tangents meeting in D, *ultimately*, when C comes up to A, $DC = DA$.

Draw a subtense BD, and AV, CX the chords of curvature parallel to it. Then by the last Cor.

$$AV.BD = DA^2,\ CX.BD = DC^2.$$

Therefore
$$\begin{aligned} DC^2 : DA^2 &:: CX.BD : AV.BD \\ &:: CX \quad : AV \end{aligned}$$

and therefore ultimately $DC^2 : DA^2$ are in a ratio of equality, and $DC = DA$.

SCHOLIUM.

ON POINTS WHERE THE CURVATURE IS NOT FINITE.

Definition. Fig. 10. If AB, AC be two curves having a common tangent AD; and if DCB be the subtense; the ratio of the angles of contact BAD, CAD of the curves AB, AC, is measured by the *ultimate* ratio of the subtenses BD, CD.

Lemma *A*. *If the subtense* DB *be as* AD^2, *and* DC *as* AD^3, *the angle of contact of* AC *is infinitely less than that of* AB.

For let dbc, parallel to DCB, be drawn; then

	$DB : db :: AD^2 : Ad^2$ by hyp.
Whence	$DB^3 : db^3 :: AD^6 : Ad^6$
also	$DC : dc :: AD^3 : Ad^3$ by hyp.
whence	$DC^2 : dc^2 :: AD^6 : Ad^6$
therefore	$DB^3 : db^3 :: DC^2 : dc^2$
and	$DB^3 : db^3 :: DC^3 : DC.dc^2$
or	$DB^3 : DC^3 :: db^3 : DC.dc^2$.

Let Ad, and therefore db, dc, remain finite, and let AD be indefinitely diminished. Then *ultimately* DC vanishes, and the ratio $db^3 : DC.dc^2$ becomes infinitely great. Therefore the ultimate ratio of $DB^3 : DC^2$ is infinitely great: and therefore so also is the ratio of $DB : DC$, and that of the angles of contact of AB, AC.

LEMMA *B*. Fig. 10. *If the subtense* DB *be as* AD^m *and* DC *as* AD^n, *where* m *is less than* n, *the angle of contact of* AC *is infinitely less than that of* AB.

For because

$$\begin{array}{lllll} & DB : db & :: AD^m : Ad^m, & DB^n : db^n :: AD^{mn} : Ad^{mn} \\ \text{so} & DC : dc & :: AD^n, : Ad^n, & DC^m : dc^m :: AD^{mn} : Ad^{mn} \\ \text{hence} & DB^n : db^n & :: DC^m : dc^m. & \text{Multiply by } DC^{n-m} \\ & DB^n : db^n & :: DC^n : DC^{n-m}dc^m & \\ \text{or} & DB^n : DC^n & :: db^n : DC^{n-m}dc^m. & \end{array}$$

When D approaches indefinitely to A, DC vanishes, and therefore DC^{n-m} vanishes, because $n-m$ is positive; but db, dc remain finite. Therefore the ratio $DB^n : DC^n$ becomes infinite; and therefore the ratio $DB : DC$, and that of the angles of contact.

COR. Hence if DB be made successively to vary as AD^2, AD^3, AD^4, AD^5, AD^6, AD^7, &c. we shall have an infinite series of angles of contact each infinitely less than the preceding; and if we take DB successively as AD^2, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, $AD^{\frac{7}{6}}$, &c. we shall have another infinite series of angles of contact, of which the first is of the same kind with the angles of contact in circles (Lemma XI), the next infinitely greater, and each angle infinitely greater than the preceding. Moreover between any two of these angles we may insert a series of mean terms, infinite in each direction, of which each is infinitely less or infinitely greater than the preceding one. Thus between the terms AD^2 and AD^3 we may insert the series $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{17}{6}}$ &c. And again, between any two of the angles of this series, we may insert a new series of intermediate angles, differing from

each other by infinite intervals. And there is no limit to this progression of properties.

DEF. When the *circle of curvature* is infinitely *large*, the *curvature* is considered as infinitely *small*, and conversely.

LEMMA *C*. Fig. 11. *If the angle of contact of* AC *be infinitely less than that of* AB, *the diameter of the circle of curvature of the former curve will be infinitely greater than that of the latter.*

Let BCD be perpendicular to AD, BG to AB, CH to AC. Then $AG.BD = AB^2$, $AH.CD = AC^2$. And *ultimately* $AB^2 = AC^2$, therefore ultimately $AG.BD = AH.CD$; or ultimately $AG : AH :: CD : BD$.

If therefore CD be infinitely less than BD ultimately, AH will ultimately be infinitely greater than AG: that is, the diameter of curvature of AC will be infinitely greater.

COR. 1. The curvature of a curve is infinitely less than that of another curve, when the circle of curvature is infinitely greater. Hence propositions concerning the curvatures agreeing with those already proved concerning the subtenses (Lemma *B* and Cor.) are true.

COR. 2. The curvature of a circle and of any conic section is finite. Hence the 11th Lemma is true of all points of such curves; but not of points of curves where the curvature is infinitely greater or infinitely less than that of a circle.

Subsection II. DYNAMICAL *Definitions and Principles.*

Dynamics is the science in which we determine, for any body or collection of bodies, the motions produced by given forces, or the forces which produce given motions. The general problem is to investigate the relation of time, space, velocity and force for any point of a system which is in motion under given circumstances.

The notions of *time*, *space*, *motion*, are taken for granted. Motion, except when the contrary is expressed, is understood

to be absolute motion, that is, motion considered with reference to the parts of fixed space.

Motion may be quicker or slower: that is, the moving body may pass over a larger or smaller space in the same time. The *velocity* is the measure of the degree in which the motion is quick or slow.

DEFINITION AND MEASURE OF VELOCITY.

When a body always passes over equal spaces in equal times, both great and small, its motion is said to be uniform, and its velocity is *constant*.

In this case the velocity is measured by the *space passed over in a second* (or other unit of time).

If we take s for the space in this case passed over in t seconds, v for the constant velocity, s will be equal to tv, by the supposition. Therefore $v = \frac{s}{t}$: and hence the velocity of a body moving uniformly may be found by dividing the space passed over in any time by that time. This quotient will be constant, whatever be the time.

When the motion is not uniform, the quotient of the space described in any time beginning at a certain point, is not constant for different times.

In this case the velocity is *variable;* it is measured by the *ultimate value* or *limit* of the quotient of the space by the time.

For the measure of the velocity at any point must have these two properties:—it must become the quotient of the space in any time divided by the time, when the motion becomes uniform;—and it must be deduced solely from the condition of the motion *at* the given point to which the velocity is attributed; it must not be affected by the conditions of the motion after the body has proceeded to another point.

Now the *limit* of the quotient of the space from the given point divided by the time, possesses these properties, and no other quantity does. For when the motion is uni-

form, this quotient is constant, and therefore is equal to its ultimate value, and this ultimate value rightly measures the velocity. And in all cases the limit depends only upon the condition of the motion at the given point; for as we approach the limit we diminish indefinitely the effect of any change which takes place in the motion after the given point.

Hence $v = \text{limit of } \dfrac{\text{increment of } s}{\text{increment of } t}$.

COR. 1. By the principles of the Differential Calculus, if s and t be exprest either one in terms of the other, or by means of any other analytical connexion, we have

$$\text{limit of } \frac{\text{increment of } s}{\text{increment of } t} = \frac{ds}{dt}: \text{ therefore } v = \frac{ds}{dt}.$$

COR. 2. If v be the velocity at any point, t any time from that point, s the space in that time; then v = *ultimate* value of $\dfrac{s}{t}$.

Let s' be the space which would have been described in the time t, if v had continued constant, then $v = \dfrac{s'}{t}$.

Hence $\dfrac{s'}{t}$ = *ultimate* value of $\dfrac{s}{t}$, when t is indefinitely diminished.

And therefore $s : s'$ is *ultimately* as $1 : 1$.

COR. 3. Fig. 12. If a rectangle be taken of which one side (MN) represents t, and the other (MP) v, the rectangle ($MPUN$) will represent s': for $s' = tv$.

COR. 4. Fig. 12. If a curve be taken of which the abscissas represent the times and the ordinates the velocities, the curvilinear areas will represent the spaces described.

Let the ordinates of the curve BC from D to E represent the velocities, at the times represented by abscissas from AD to AE: the area $BDEC$ will represent the space described in the time DE.

Take MN a portion of the line of abscissas; and if $MPQN$, the corresponding portion of the area, do not represent the corresponding space described, let $MRSN$ represent this space. Now by Cor. 2, $MPUN$ represents the space which would have been described if the velocity had continued constant from M. Hence by Cor. 1, $MRSN$ and $MPUN$ are *ultimately* equal, when MN is indefinitely diminished. But $MRSN$ and $MRVN$ are *ultimately* equal: therefore MU and MV are *ultimately* equal. But this cannot be, except R coincide with P, since MU and MV are in the ratio $MP : MR$. Hence the curve of which the area represents the space described, passes through P. For a like reason it passes through every point of $BPQC$, and therefore is that curve itself. Hence the areas of the portions of the curve $BPQC$, cut off by ordinates, represent the spaces described in times corresponding to the abscissas.

DEFINITION AND MEASURE OF ACCELERATING FORCE.

Force is that which produces or tends to produce motion. *Accelerating force* is that which produces increase (or diminution) of *velocity*. It is greater or less as the increase of velocity in a given time is greater or less.

When the increase of velocity in equal times, both great and small, is always equal, the motion is said to be uniformly accelerated, and the force in the direction of the motion to be *constant*.

In this case the force is measured by the *velocity added in a second* (or other unit of time).

If we take v for the velocity added in t seconds, f for the constant force, $v = ft$ by the supposition. Therefore $f = \frac{v}{t}$: whence the accelerating force in the case of a body uniformly accelerated, is found by *dividing* the velocity added in any time, *by* the time. This quotient will be constant, whatever be the time.

When the motion is not uniformly accelerated, the quotient of the velocity added after any point, divided by the time in which it is added, is not constant for different times.

In this case the force is variable: it is measured by the *limit* of the quotient of the velocity added divided by the time.

If v be the velocity at any instant, v' the velocity after a short time t, f the force, $f =$ ultimate value of $\frac{v' - v}{t}$, when t is indefinitely diminished.

For the measure of the accelerating force at any given point must become the quotient of the velocity generated, by the time in which it is generated, when the motion becomes uniformly accelerated motion: and it must depend only upon the conditions of the motion *at* the given point and at no other. The limit of the quotient of the velocity generated, divided by the time, is the only quantity which possesses these properties; and this limit therefore is the measure of the accelerating force.

Cor. 1. The accelerating force at any point is measured by the velocity which would have been generated in one second, if the force had continued constant from that point.

Cor. 2. The velocity added in any time from a given point is *ultimately* equal to the velocity which would have been generated in the same time if the force had continued constant from the given point: the time being indefinitely diminished.

Cor. 3. Fig. 12. If the force be always finite, though variable, the curve BC in the last proposition will cut the line of abscissas AE at a finite angle.

Let A represent the point of time at which the motion commences. Let LO, MP, NQ be ordinates, PU parallel to AD. Then by the definition, the force at A is the *ultimate* value of $\frac{OL}{AL}$ when AL vanishes; and the force at P is the *ultimate* value of $\frac{QU}{MN}$ when MN vanishes.

Since the force is always finite, its values at different points of time must have a finite ratio. But the ultimate value of $\frac{QU}{MN}$ or $\frac{QU}{PU}$ is necessarily somewhere finite, for otherwise the curve would coincide with the line AE: therefore it is always finite: therefore the ultimate value of $\frac{OL}{AL}$ is finite. But, excepting the case in which $\frac{OL}{AL}$ ultimately vanishes, (in which case AL is a tangent at A,) $\frac{OL}{AL}$ is ultimately equal to $\frac{KL}{AL}$, AK being a tangent at A. Hence $\frac{KL}{AL}$ is finite, and AK makes a finite angle with AE. Therefore the curve cuts the line of abscissas AE at a finite angle in A.

Lemma X. *If a body be urged by any finite force, the spaces described from the beginning of the motion, when they are taken indefinitely small, are* ultimately *as the squares of the times.*

Fig. 5. Let the times be represented by AD, AE, and the velocities generated by the ordinates DB, EC. Then the spaces described will be as the curvilinear areas ADB, AEC: (Cor. 4 to Definition of Velocity:) and the curve will cut the abscissa at a finite angle, because the force is finite (Cor. 3 to Definition of Accelerating Force.) Therefore, by Lemma IX, the spaces ADB, AEC are *ultimately*, when D and E approach indefinitely to A, as the squares of AD, AE. Therefore the spaces described in the times AD, AE, are *ultimately* as the squares of the times.

Cor. 1. If the force be constant, the velocity DB increases proportionally to the time AD, and ABC is a straight line.

Cor. 2. Hence the space described in any time by the action of a constant force from rest, is equal to half the space described in the same time with the velocity last acquired.

For the space described in the time AE is represented by the triangle AEC, while the space described in the time AE, with the last acquired velocity EC, is the rectangle $AE \times EC$, which is double the triangle.

Cor. 3. When the force is not constant, the space described from the beginning of the motion is *ultimately* half the space described in the same time with the last acquired velocity.

For both these spaces are ultimately equal to what they would have been had the force been constant.

Cor. 4. If f represent a constant force, measured by the velocity which it would produce in a time 1, ft will be the velocity produced in time t, and therefore, by Cor. 2, $\frac{1}{2}ft^2$ will be the space described from each in a time t.

Cor. 5. Since s the space $= \frac{1}{2}ft^2$, and $v = ft$, $v^2 = f.ft^2$, whence $v^2 = 2fs$.

THE LAWS OF MOTION.

Law 1. *A body in motion, not acted on by any force, will move on in a straight line, and with a uniform velocity.*

First, it will move in a straight line. For if it do not, it must move in some curve; and external circumstances must determine towards which side the convexity must lie, and how great the curvature must be. But, when a body's motion is influenced by external bodies, those bodies are said to exert force upon it, which is contrary to the supposition. Hence, a body influenced by no force, cannot describe any path but a straight line.

Next, it will move with a uniform velocity. As we remove the known causes which retard a body's motion, we find that we remove the retardation, and this without limit; so that it is evident, that if we could entirely remove the external causes of retardation, the body would not be retarded at all. There is no internal principle which tends to diminish the velocity.

The common causes by which motions are retarded, and finally stopped, are friction and the resistance of the air. If a wheel turn on a very smooth axle, it will revolve for a long time; and the longer, as we remove more of the friction by making the axle smoother; and if we also diminish the resistance of the air by making the wheel revolve in an exhausted receiver, the motion will continue still longer. We can never quite remove the friction or the resistance; and it is on that account, that the rotation cannot be made to continue for ever without diminution.

The remaining retardation is always such as is duly accounted for by the remaining resistance.

Hence the first law of motion is true.

Law 2. *When any force acts upon a body in motion, the* change *of motion which it produces is the same, in magnitude and direction, as the effect of the force upon a body at rest.*

Both the original motion, and the change of motion communicated, are retained in their own directions. Thus, in fig. 13, if the body be in motion with a velocity which would carry it through AB, and be acted on by a foroe which would carry it through AC in the same time, it will at the end of that time be found at the point D, $ACBD$ being a parallelogram.

The proofs of this law are of the following nature.

The relative motions of bodies with regard to a limited space are the same, whether that space be at rest, or move uniformly in a straight line: (all angular motion of the space being excluded.) And the differences of the velocities of bodies moving in the same directions, and the sum of the velocities of those which move in opposite directions, are the same on each supposition. Hence the congresses, and impacts of the bodies, and all the mutual actions which depend on their relative velocities, will be the same on each supposition. And the velocities arising from these mutual actions will be compounded with the velocity of the space, and if this second law be true, will produce resulting re-

lative motions, which will be the same as if the space were at rest. And this agrees with experiment. For the effects of the impacts and other mutual actions of bodies are the same in a ship in uniform and steady motion, as if the ship were at rest.

Also the effects of gravity in altering the relative motions of bodies included in a limited space, are the same whether the space be at rest, or be moving uniformly in a straight line. Hence the effect of gravity consists in compounding with the previous motion of the body, the motion which the same gravity would communicate to a body at rest.

Thus a body let fall from the top of the mast of a vessel in motion, will fall down along the mast, (if vertical;) thus retaining the horizontal motion of the ship, as well as the motion communicated by gravity.

A body thrown across the deck by a person on board, will in the same manner proceed in the direction in which it is thrown relatively to the vessel; thus, both retaining the motion of the vessel, and obeying the force by which it is projected.

Since the earth revolves about its axis, every part of the earth's surface is in motion. This motion is circular, but the radius of the circle in latitude 60°, being 2000 miles, and in lower latitudes larger still, the motion may be considered rectilinear in all experiments on the mutual action of bodies. And the motions impressed on bodies by any forces, are compounded with the motions which the bodies previously possess, in consequence of the rotation of the earth.

Now the motions impressed on a body by the same agent, are the same, whatever be their direction with respect to the direction of the earth's motion. Thus a pendulum oscillates in the same time swinging east and west, or north and south.

Also the motions impressed on bodies in different parts of the earth, are the same, relatively to the earth, if the forces be the same; thus shewing, that besides the motions

impressed, the bodies retain the motions of the parts of the earth where they are without change. And the velocities of the earth in different places are very different.

Hence the composition of the velocity impressed, with the velocity previously existing, is true when the two directions make any angles whatever, and for an infinite number of velocities.

Therefore the second Law of motion is true.

Cor. 1. Fig. 13. Let AD be the space which would be described in any time in consequence of the velocity; AB the path which is actually described in the same time in consequence of the action of a variable deflecting force; AC the space through which the force, retaining the magnitude and direction which it has at A, would cause a body to move from rest in the same time; Db equal and parallel to AC: then will DB be *ultimately* equal to Db, and coincident with it in direction.

Cor. 2. Fig. 13. Hence if in any two curves, we take arcs, as AB, $a\beta$, described in equal times; the subtenses of the angles of contact of these arcs, drawn parallel to the directions of the forces, will ultimately be as the forces at A, a.

Cor. 3. Forces tending to centers are to the force of gravity, as the subtenses of the arcs of the curves, to the vertical subtenses of the parabolic arcs described by projectiles in the same time.

Cor. 4. If any number of bodies move in any manner (acting on each other or otherwise) and if they be all acted upon by equal accelerating forces acting in parallel lines, they will continue to move in the same manner, relatively to each other, as if these forces had not acted.

For, whatever be the previous motions and forces, in virtue of the action of the new forces all the bodies will be equally displaced in the same direction; (by Law 2;) and therefore their relative positions at any moment, and their relative motions, will continue unaltered.

COR. 5. Fig. 14. If DE be taken, representing the velocity generated by the force, in the time in which the body describes AC; and AD, representing the velocity at A; AE will *ultimately* be parallel to the direction of the body's velocity at the end of that time.

Draw CB a tangent at C. Then by Lemma XI, Cor. 2 AB, BC are ultimately equal: and BC, BD are ultimately equal; therefore AB, BD are ultimately equal, and $AD = 2\,AB$ ultimately.

Also by Lemma X, Cor. 3, DE is ultimately $2\,DC$. Therefore ultimately $AD : BD :: ED : CD$; and AE is parallel to BC.

COR. 6. The *magnitude* of the velocity at C is also properly represented by AE.

COR. 7. Hence in ultimate arcs, if the velocities with which bodies are moving, be *compounded* with the velocities which the extraneous forces would generate, according to the laws of the *composition* of statical forces, we shall have the resulting velocities.

COR. 8. If the force be supposed to act instantaneously, the same will be true. Hence if BE, Fig. 15, be the velocity of a body at B, and EC the velocity communicated instantaneously at B, BC will be the direction and velocity of the motion from B.

MEASURE OF THE QUANTITY OF MATTER.

Bodies are considered as having the *same* quantity of matter when they produce, by their matter, the same mechanical effects. The effect which is more peculiarly taken as the measure of this quantity, is the resistance to motion, or rather, the smallness of the motion produced by a given pressure. The quantity of matter is supposed to be greater, exactly in proportion as the velocity, or accelerating force, resulting from a given pressure upon the body, is smaller.

The resistance to motion measured as above is called the *inertia*. Hence the quantity of matter is the same when the inertia is the same.

In bodies of the same material, the quantity of matter is proportional to the magnitude; and it will appear in proving the third Law of Motion, that the inertia is also proportional to the magnitude: hence in such bodies, the inertia is as the quantity of matter.

Two bodies of different materials are supposed to have the same quantity of matter, when they have the same inertia. And hence, combining this with what has just been said, the quantities of matter of all bodies are as the inertia.

It is found by experiment that the inertia of different terrestrial bodies of the same, or of different materials, is proportional to their weight at the same place. This appears from the experiments by which the third Law of Motion is proved: for the quantities of matter in those experiments are compared by comparing the weights; and on this supposition the law is found to be true.

The inertia of a body cannot be so easily determined by direct trial as the weight can. It is therefore convenient to compare the quantity of matter, or *mass*, of different bodies, by assuming it to be proportional to their weights at the same place: and this is supposed to be done in the following reasonings.

MEASURE OF MOVING FORCE.

Moving Force is measured by the Accelerating Force multiplied into the Quantity of Mattter.

Cor. 1. The moving force is as the accelerating force multiplied into the quantity of matter, and the accelerating force is as the velocity communicated in a given time. Hence the product of the velocity in a given time into the quantity of matter will be as the moving force.

Cor. 2. The product of the velocity and quantity of matter is called the *Momentum:* Hence the momentum communicated in a given time is as the moving force.

Law 3. *When pressure communicates motion directly* (that is, in the direction of the pressure,) *the moving force is as the pressure.*

By Cor. 2, to the Def. of Moving Force, it appears that this law will be proved, if it is shewn that the momentum communicated in a given time is as the pressure which communicates it.

This may be exemplified in various ways. When two bodies P, Q are suspended over a pully, if P be the heavier, $P-Q$ is the portion of P which is employed in producing motion: and if we neglect the pully, as of small weight, $P+Q$ is the mass moved. Hence, if the law be true, $P+Q$ multiplied into the velocity generated in a given time will be proportional to $P-Q$. And this is found to agree with experiments.

Atwood's machine is an exemplification of the same kind: but in this machine the inertia of the pullies, and the friction, are considerable, and require to be taken into the account. But the momentum generated in a given time is always found to be proportional to the weight by which the machine is put in motion.

See Atwood *On Rectilinear and Rotatory Motion*, Sect. 7: also Mr Smeaton's experiments, *Phil. Trans.* Vol. LXVI.

The mutual pressures of two bodies against each other are in all cases equal. The same is true of the mutual force of attraction, which may be considered as a pressure. Hence, if Law 3 be true, the momentum gained by one body and that lost by the other, in their direct action, ought to be equal. It is found in all cases to be so.

Impact is really a pressure of short duration. Hence the momentum gained and lost by the two bodies in direct impact must be equal, if this law be true. It is found to be so. See Newton, *Scholium to the Laws of Motion*, Prin. B. I.

When the pressure of a weight $P-Q$ produces motion in a mass $P+Q$, if we suppose Q to become 0, we have the case of a body falling freely; in this case by the third Law, the accelerating force is as $\frac{P}{P}$, that is, it is independent of P.

And such is found to be the case in reality; when bodies fall in free space, they fall with equal velocities, whatever be their weights. The truth of this proposition was first established experimentally by Galileo, in opposition to the doctrine previously prevalent, that heavier bodies fell more quickly in proportion to their weights. It is true, that in the air heavier bodies do fall somewhat more quickly, *cæteris paribus*, than lighter ones; but this is owing, not to any difference in the accelerating force arising from their weight, but to the resistance of the air to their motion, which is, *cæteris paribus*, greater in the case of the lighter bodies.

The velocities of bodies falling freely, are too great to be easily measured with accuracy: but if we suspend a weight at the lower end of a string of which the upper end is fixed, the force which causes this pendulum to oscillate has a certain relation to the weight of the body; and may be calculated by knowing the length of the pendulum. Thus it appears (by the resolution of pressures) that at equal small distances from the vertical line, the pressure, or force, which urges the pendulum towards the vertical line, is (ultimately) inversely as the length of the pendulum. Hence, supposing the accelerating force to be proportional to the pressure, it appears (Lemma x. Cor. 5) that the time of reaching the vertical line will be as the square root of the length of the pendulum; and the time of an oscillation is the double of the time just mentioned. Therefore in different pendulums the time of oscillation will be as the square root of the length of the pendulum, if the third law of motion be true.

Now such is found to be really the case. And this is an observation which admits of great accuracy: for the oscillations of pendulums being perpetually repeated, any deviation in the duration of their oscillation from that given by theory, is also perpetually repeated, and will thus in the course of time become sensible, however small it be.

Let there be a seconds pendulum, and another four times as long, which ought therefore to swing double seconds. If its

real time of oscillation deviates from this by $\frac{1}{1000}$ of a second, in 2000 seconds (that is in little more than half an hour) it will be wrong a whole oscillation, and will be swinging from right to left during two certain seconds, when it ought to have been swinging from left to right by the Law of Motion.

WEIGHT AND ATTRACTION.

For a given material, the weight of a body, or its tendency to the earth, is proportional to the magnitude. By connecting two bodies, we obtain a weight equal to the sum of the two weights; and it is found that the weight of a body is independent of the figure which its materials are made to assume. Hence in such cases, the weight is as the quantity of matter.

Bodies of different materials are supposed to have the same quantity of matter when they have the same inertia. And if the weight of such bodies be proportional to the inertia, in this case also, *the weight is proportional to the quantity of matter.*

This is proved to be true by observation and experiment. Bodies of different materials, as well as of different magnitudes, falling freely, descend with the same velocity in the same time. And in this case also more accurate experiments may be made by means of pendulums.

Newton made experiments with pendulums for this purpose, (Principia B. III. Prop. VI.) of which he gives the following account. "I experimented with gold, silver, lead, glass, sand, common salt, wood, water, wheat, in the following manner. I took two round and equal boxes: I filled one with wood, and I suspended the same weight of gold, as exactly as I could, in the center of oscillation of the other. These boxes, hanging by equal strings of eleven feet long, were pendulums precisely equal, in respect of weight, figure, and the resistance of the air. Being made to oscillate near each other, they swung to and fro, in exact agreement, for a very long time. Therefore the quantity of matter or inertia in the gold, was to the quantity of matter in the wood,

as the force producing motion in the gold, to the force producing motion in the wood; that is as the weight of the gold to the weight of the wood: and in like manner with regard to the other subtances. In bodies of the same weight, a difference of the relative quantity of matter amounting to less than $\frac{1}{1000}$ of the whole, might be clearly detected by this method."

It was proved by Newton that all the bodies of the universe attract each other with forces of the same kind as that with which the earth attracts the bodies that fall to it. And it appears that in the case of this attraction also, the force or pressure arising from the attraction is proportional to the quantity of matter (or inertia) of the attracted body.

This proposition also is to be proved from observation. It will be true if it is found that the *accelerating force* upon all bodies at the same distance from the attracting body is equal; for the accelerating force is as $\frac{\text{pressure}}{\text{inertia}}$ by the third Law of motion: and hence, if this be always the same, the pressure is as the inertia.

In the cases which have been most accurately calculated this is found to be true. The earth attracts the moon with the same accelerating force with which it would attract a mass of stone or iron at the same distance: that is, with the same force as that which acts upon terrestrial bodies, diminished in the ratio of the square of the distance from the earth's center. The sun attracts both the earth and moon, and thus disturbs the moon's motion round the earth: and it is found by calculation that these perturbations are such as would occur, supposing the sun to attract the earth and moon *equally*, (for equal masses,) at the same distance. In like manner the planets attract each other, and thus disturb each other's motions, and the perturbations are calculated on the supposition that the accelerating force of each planet is the same, at the same distance, on all the other planets.

It has sometimes been supposed that the effect of an attracting body upon other bodies is not exactly proportional

to the inertia of the attracted bodies, but may depend upon their nature and constitution; this point is to be decided by observation. The attractive force of Jupiter upon Juno or Pallas may be found to be different from his attractive force upon Saturn, by finding that the perturbations calculated on the supposition of the same attractive force (or *mass*) in the two cases do not rightly represent the observations.

Assuming the Newtonian doctrine, that all bodies attract each other with forces proportional to their quantities of matter, experiments have been made for the purpose of determining the quantity of matter of the earth, by comparing its attraction with that of known bodies. Dr Maskelyne ascertained by observation how far the attraction of a known mountain (Shehallien) drew the plumb line aside from its direction. Mr Cavendish made experiments in order to determine the mutual attraction of leaden balls. From each of these sets of observations was deduced the mass of the earth; and hence (its bulk being known) its density with respect to water.

The density, according to the observations of Dr Maskelyne, is nearly 5; according to the experiments of Cavendish it is 5,3; taking the calculations of Dr Hutton. The near agreement of these results, in experiments which do not admit of great accuracy, shews that the attraction exerted by the materials of Shehallien upon the plumb line, and the mutual attraction of the leaden balls, is very nearly as their weight, that is, as the attraction of the earth upon the different masses: for the weight was taken as the measure of the quantity of matter of the masses in calculating the earth's density.

The weight of bodies at a given place is proportional to the inertia or quantity of matter of each. If we go to another place at which the force of gravity is different, the weights of the bodies will be different. But it is found that the accelerating forces of different bodies, arising from their weight, which are equal at one place, are equal at any other place, however different the bodies be. Hence it appears that the weight of bodies depends only upon the inertia or quantity of matter in each, and on the force of gravity at

the place where they are: and the magnitude of the latter element is common to all bodies, and has not peculiar values relatively to bodies of different kinds.

ON THE APPLICATION OF THE LAWS OF MOTION.

When pressure communicates or changes motion by acting *obliquely*, the effect will be found by combining the second and third Laws of Motion.

When pressure acts upon a body in motion, the result will be found by compounding, according to the laws of statical forces, the momentum which the body already possesses with the momentum which the pressure would generate in the same time.

For the resulting velocity was found by compounding the velocities according to this rule; Law 2, Cor. 5; and hence, if we multiply each velocity by the mass of the body, the resulting momentum will be found by the same rule.

The mutual pressures of any number of bodies in motion, connected in any manner, are determined by the laws of statics, the system being considered as a machine, and the pressures as forces which keep each other in equilibrium on the machine.

Hence the motions of any number of bodies will be such as result, when we combine with the momenta which the bodies already possess, the momenta which their mutual pressures communicate to them: and the mutual pressures will depend on each other by the laws of statics.

This rule will be further explained under the name of *D'Alembert's Principle.*

The three Laws of Motion are thus proved by experiment; and these are the only mechanical principles which we require in order to calculate the motions of any bodies whatever, acting upon each other in any manner.

The motions of various machines and material systems, some simple and some complicated, of the most diverse forms

and acted upon by various kinds of forces, have been calculated upon the preceding principles. The results of calculation have been found to agree with the results which occur in fact, to as great a degree of nicety as has been found attainable in observation. By this agreement, the truth of the laws which are made the foundation of this calculation, is established beyond the possibility of a doubt.

ON THE PROOF OF THE LAWS OF MOTION.

The three Laws of Motion are proved from observation and experiment, and no other kind of proof is applicable in this case. Arguments derived from abstract considerations are sometimes given as demonstrations of these laws: but in all these cases there is some fallacy in the reasoning. A few such may be mentioned.

LAW 1. It is sometimes asserted to be manifest that a body left to itself can neither accelerate nor retard its own motion, and that therefore the velocity will continue constant.

If it be meant that a body so circumstanced cannot exert any accelerating or retarding *force* on its own motion, this is true, because such force is something extraneous: but in this sense it does not follow that the velocity will remain constant: for the thing to be proved is that this will not change without the action of extraneous force. If it be meant that a body so circumstanced can neither increase nor diminish its own *velocity*, this is neither obvious nor certain, without reference to experiment.

This Law is sometimes attempted to be proved in the following manner. The velocity will not change; for if it change, that change must be according to some function of the time: as $v = c + at^m + bt^n + \&c.$ Now there is no condition involved in the nature of the case by which the coefficients, a, b, &c. can be determined to be of any one magnitude rather than of any other. Hence there cannot be such coefficients; therefore $a = 0$, $b = 0$, &c. and $v = c$.

This is fallacious; for the question is not what is the law of the velocity of a body *from the nature of the case*, that is, from our definitions, but what it is *in fact*.

There would be no contradiction in supposing the velocity to decrease perpetually in a geometrical ratio, as the time increases. Thus let $v = \frac{c}{m^t}$, where t is measured in seconds. If m were $\frac{10}{9}$, the velocity would in each second lose $\frac{1}{10}$ of the value which it had at the beginning of that second*.

LAW 2. The object of this law is to give a rule for the combination of the effect of a *force* acting on a body, with the *velocity* which the body already possesses. Hence it is not properly proved, either by the combination of two velocities, or two forces; though propositions relating to such combinations are sometimes offered as proofs of this law.

The combination of two velocities represented in magnitude and direction by the two sides of a parallelogram, will produce a resulting velocity represented by the diagonal. Hence it is sometimes said, that *if* a body moving with one of these two velocities *have* the other communicated to it, it will describe the diagonal. But the object of the second law is to show that the body *will have* the second velocity communicated to it, this velocity being that which the force would produce in the body if at rest.

It is sometimes said that the action of a force parallel to the line *DB* cannot accelerate or retard the approach of the body to the line *DB*. But this is not obvious, except we suppose the body to retain the two velocities independently: whereas in reality, the two velocities are confounded into a single velocity, and the dependence of this on the two in nature cannot be known from our definitions merely.

* In the case when $v = \frac{c}{m^t}$ we have $\frac{ds}{dt} = \frac{c}{m^t}$; and $s = b - \frac{c}{m^t \, \mathrm{l}m}$. When $t = 0$, c is the velocity; and when t is infinite, the space described is $\frac{c}{\mathrm{l}m}$. If $m = \frac{10}{9}$, $\mathrm{l}m = .105$, and the whole space described is $9\frac{1}{2} \times c$ nearly.

It is sometimes said that the velocity *DB* may be produced by a certain force, which force will be proportional to the velocity, and may be substituted for the velocity; and then, this force being compounded with the force which acts in the direction *AD*, by the laws of statics, the result will be a force in the direction *AB*, and a motion in that direction.

To this reasoning the following objections exist; 1. It does not follow because the velocity *DB* may be *produced by* a certain force, that we may therefore *substitute* the force for the velocity. 2. The force here spoken of is an impulsive or instantaneous force; but we are in no way justified in applying to such forces the proposition concerning the statical composition of forces of a kind quite different. 3. There are no forces physically instantaneous; impact being really a short pressure. 4. If we suppose instantaneous forces, we have no right to assume that they are governed by the same rules as physical forces, in the action of which time is a necessary element. 5. If we attempt to substitute physical forces, acting in time, for instantaneous forces, in this proof, the whole reasoning becomes inapplicable.

LAW 3. The third Law is sometimes understood to be asserted in the expression that "action and reaction are equal and opposite." This assertion expresses a truth of statics, where action and reaction denote statical forces, namely, pressures in equilibrium. To make this assertion include the third Law of motion, we must give to *action* and *reaction* entirely new meanings, namely, the momenta gained and lost. That these momenta are proportional to the pressures, is the part of the law which requires a proof from experience.

In some treatises on Dynamics, and especially in French writers, the second and third Laws are both understood to be included in the Proposition that "the force is as the velocity." This Proposition is then taken in two different senses. In order that it may express the third Law, it means that the velocity directly communicated to a given body is as the pressure which *communicates* it. In order that the proposition, that "the force is as the velocity," may express the

second Law, it means that the velocity which a body has may be *replaced* by a force proportional to it, and that this force, and the extraneous forces which act on the body, may be compounded according to the laws of statics. These are two different assertions, and require separate proofs.

The third Law determines what is the amount of the motion which a given force will produce: the second Law determines in what manner this motion will be compounded with a previously existing motion.

The second Law of motion is proved by the experiment suggested by Laplace; namely, by the fact that the oscillations of the same pendulum are equally quick in all azimuths. The third Law is proved by experiments which shew that the time of oscillation is proportional to the square root of the length of the pendulum. One of these laws cannot be deduced from the other, except one of these facts can be shewn to be implied in the other.

SECTION II.

GEOMETRICAL SOLUTION OF THE DIRECT PROBLEM OF CENTRAL FORCES.

[Newton. Principia. *Book I. Section II.*]

The direct problem of central forces is, to find the force, tending to a given centre, which will cause a body to describe a given curve. The motion is supposed to take place in non-resisting spaces.

Prop. I. ***When a body moves in a curve, acted on by forces tending to a fixed point, the areas which it describes by rays drawn to the centre of force, are in a constant plane, and are proportional to the times.***

Fig. 16. Let the time be divided into equal portions, and in the first portion let the body describe AB. By the first law of motion, if no force were to act on the body, it would in the second portion of time go on to c, in the same straight line, describing Bc equal to AB. But when the body comes to B, let a force tending to the centre act on it by a single instantaneous impulse, and turn the motion in the direction BC. Draw cC parallel to BS; and by the second law of motion, the body will describe BC in the second portion of time, C being in the plane ASB. Join SC; the triangle SBC is equal to SBc, because cC, BS, are parallel; and therefore to SAB, because Bc is equal to BA.

In like manner if a centripetal force towards S act impulsively at C, D, E, &c. at the end of equal successive portions of time, causing the body to describe the straight

lines CD, DE, EF &c.: these lines will all lie in the same plane, and the triangles SCD, SDE, SEF will all be equal to SAB and SBC. Therefore these triangles will be described in equal times, and in a constant plane. And we shall have

$$SADS : SAFS :: \text{time in } AD : \text{time in } AF.$$

Let now the number of the portions of time in AD, AF, be augmented, and their magnitude diminished, indefinitely; and the above proportion will still be true.

Therefore also *ultimately* $SADS : SAFS ::$ time in AD : time in AF.

But *ultimately* the polygon $ABCDEF$ &c. becomes a curve line, and the force which acted impulsively at B, C, D, E, &c. becomes a force which acts continuously at all points*.

Therefore in this case also we have curvil. areas $SADS$: $SAFS$ :: time in AD : time in AF.

Cor. 1. The velocity of a body attracted to a fixed centre, is inversely as the perpendicular from the fixed centre upon the tangent to the curve.

For the velocities in the polygon at two points A, D are as AB, DE, because these lines are described in equal portions of time. But if SY, SZ be perpendicular on these lines, $SY . AB = SZ . DE$, because the triangles are equal.

$$\text{Therefore vel. at } A : \text{vel. at } D :: SZ : SY.$$

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* The force in the polygon is a force which acts instantaneously at the angles, and thus produces a motion composed of straight lines. The force in the curve acts continuously. The force at each angle of the polygon is supposed to be such as will produce, instantaneously, the velocity which the continuous forces produces in the time of describing the side of the polygon. Let AB, BC Fig. 15, be two successive sides of the polygon, $BE = BA$, and EC the space described by the action of the instantaneous force. Then if BD be a tangent to the curve ABC at B, by Lemma XI. since $BC = BA$ ultimately, $DC = DE$ ultimately, and DC is half EC ultimately. Hence by Lemma X. Cor. 3, DC, the deflection from the tangent is the space described by the action of the continuous force, as it ought to be.

And *ultimately* the velocity in the polygon becomes the velocity in the curve, and the lines AY, DZ are the tangents at A, D. Therefore, &c.

Cor. 2. All the preceding propositions are true, when the planes in which the bodies move, and the centres of force situate in these planes, are carried with a uniform and rectilinear motion. For the relative motion of P with regard to S will be the same in both cases: and the motion of P will be that which will arise from the action of S, compounded with the motion of the plane, by Law 2.

Prop. II. *If a body moves in a curve line in a constant plane, and by a ray drawn to a fixed point, describes areas, about that point, proportional to the times, it is urged by a central force tending to that point.*

The same is true if the point, instead of being fixed, be moving uniformity in a straight line.

Case 1. Fig. 16. Every body which moves in a curve, is deflected from a straight line by some force acting upon it. If the body were to describe the polygon $ABCDEF$, describing the equal triangles SAB, SBC &c. in equal times, it must at B be acted on by a force parallel to cC (Bc being equal to AB) by the second law of motion. And cC is parallel to BS, because SBC is equal to SAB or SBC: therefore the force in the polygon acts to the centre S. But *ultimately* the motion in the polygon will coincide with the motion in the curve, and the force in the polygon will be proportional to the force in the curve. Therefore in the curvilinear motion the proposition is true.

Case 2. And by Cor. 2 to Prop. 1, the force is the same, whether the plane in which the motion takes place be at rest, or move, along with the body, the figure described, and the centre of force, uniformly in a right line.

Cor. 1. In a non-resisting space, if the area is not proportional to the time, the force does not tend to the point to which the rays are drawn. It deviates to the side towards which the motion is, if the description of areas is accelerated; to the opposite side if the description is retarded.

Cor. 2. In a resisting medium also, if the description of areas is accelerated, the direction of the force deviates to the side towards which the body moves.

Prop. III. *If a body revolve about another body which is moving in any manner whatever, and if the first body describe about the second, areas proportional to the times, the first body is acted on by a centripetal force tending to the second body, and also by the whole of the force by which the second body is acted on.*

Let L be the first, T the second body. Let F be the force by which T is acted on; and let a new force equal to F, and in the opposite direction, act upon T and L in parallel lines. Then the motion of L with respect to T, will be exactly the same as before, by Cor. 4 to Law 2. But on this supposition T is acted upon by two equal and opposite forces; and is therefore in the same condition as if it were acted upon by no force; and therefore will either remain at rest, or move uniformly in a straight line. Hence by Prop. II, the body L is on this supposition acted upon simply by a force tending to the centre T. But if we now suppose L to be acted upon by the force F, in a direction parallel to that in which it really acts on T, L will be reduced to its real condition. Therefore L is really acted on by a force tending to T, and by the force F which acts on T.

Cor. 1. Hence if L describe areas about T, proportional to the times, and if from the whole force (simple or compound) which acts on L, we subtract the force which acts on T, the residue of force which acts on L, will tend to T.

Cor. 2. And if the areas which L describes be very nearly proportional to the times, the residue of the force so obtained will tend to T very nearly.

Cor. 3. And conversely if the residue of the force so obtained tend very nearly to T, the areas will be very nearly proportional to the times.

Cor. 4. If L describe about T areas which are very far from being proportional to the times, L is either acted

upon by no centripetal force tending to T, or this force is mixed and compounded with very powerful actions of other forces, besides the force which act on T. And the whole residual force on L tends to some other centre moveable or immoveable.

Since it appears that the equable description of areas about a centre, by a moving body, indicates necessarily that the main force by which the body is affected, deflected from a straight line, and retained in its orbit, tends to that centre, if the body be governed by the ascertained laws of motion; we may in any case, as for instance in the cases of the earth, moon, planets, and satellites, suppose that those centres about which the heavenly bodies describe equal areas, are the centres about which the motions of such bodies are performed in free space, according to the laws of motion.

Prop. IV. *When bodies describe different circles with uniform motions, the forces tend to the centres of the circles, and are as the squares of arcs described in equal times, divided by the radii of the circles.*

The forces tend to the centres of the circles by Prop. II.

Fig. 18. Let AB, ab be arcs described in equal times, BD, bd perpendicular to the tangents at A, a. Then DB, db are the spaces through which the bodies are deflected from the tangents, by the action of the forces to S and s; and *ultimately* are as the forces.

$$\text{Now} \quad DB = \frac{AB^2}{2AS}, \quad db = \frac{ab^2}{2as}.$$

Therefore

$$\text{force at } A : \text{force at } a :: \frac{AB^2}{2AS} : \frac{ab^2}{2as} \text{ ultimately.}$$

But if AE, ae be arcs described in any other equal times, $AE : AB :: ae : ab$, because the motions are uniform.

$$\text{Therefore} \quad AB : ab :: AE : ae : \text{and}$$

$$\text{Force at } A : \text{force at } a :: \frac{AE^2}{2AS} : \frac{ae^2}{2as} :: \frac{AE^2}{AS} : \frac{ae^2}{as}.$$

Cor. 1. The arcs AE, ae are as the velocities: hence in circles if F be the force, V the velocity, R the radius $F \propto \frac{V^2}{R}$.

Cor. 2. Also if P be the periodic time, (or time of describing the whole circle)

$$P \text{ is as } \frac{\text{circumference}}{\text{velocity}}, \text{ and therefore as } \frac{R}{V}.$$

$$\text{Therefore } F \propto R \cdot \frac{V^2}{R^2} \propto \frac{R}{P^2}.$$

Cor. 3. If P be constant, $V \propto R$, $F \propto R$.

Cor. 4. If $P \propto \sqrt{R}$, $V \propto \sqrt{R}$, $F \propto 1$, and the force is the same in all circles.

Cor. 5. If $P \propto R$, $V \propto 1$, $F \propto \frac{1}{R}$.

Cor. 6. If $P \propto R^{\frac{3}{2}}$, $V \propto \frac{1}{\sqrt{R}}$, $F \propto \frac{1}{R^2}$.

Cor. 7. If $P \propto R^n$, $V \propto \frac{1}{R^{n-1}}$, $F \propto \frac{1}{R^{2n-1}}$.

Cor 8. What has been proved of circles, is true of similar portions of similar curves having centres similarly situated. Thus in two such points of two similar curves, force in one curve (F) : force in the other (f) :: $DB : db :: \frac{AB^2}{AG} : \frac{ab^2}{ag}$. And $AB : ab :: V : v$, these being the velocities. Also, since the curves are similar and AG, ag are the diameters of curvature, $AG : ag :: AS : as$.

Hence $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$.

Cor. 9. Let V be the velocity in a circle at A, F the force, t the time of describing AB, R the radius,

$$AB = Vt, \; BD = \tfrac{1}{2}Ft^2. \quad \text{Hence } \frac{V^2t^2}{\frac{1}{2}Ft^2} = \frac{AB^2}{BD}.$$

$$\text{But ultimately } \frac{AB^2}{BD} = 2R; \text{ hence } \frac{V^2}{F} = R; \; F = \frac{V^2}{R}.$$

Cor. 10. Let P be the periodic time: π the ratio of the circumference of a circle to the diameter. Then $2\pi R = PV$,

$$4\pi^2 R^2 = P^2 V^2 \text{ and } F = \frac{V^2}{R} = \frac{4\pi^2 R}{P^2}.$$

Cor. 11. In any time T the arc described is VT; the space fallen through by force F is $\frac{1}{2} FT^2$. Now $V^2 T^2 = \frac{1}{2} FT^2 . 2R$, because $RF = V^2$. Therefore the arc described in T is a mean proportional between the space fallen through in T and the diameter.

Lemma to Prop. v. Fig. 17. *If* TXSY *be any quadrilateral figure, and if* DM, DN *be parallel* to SX, SY, *and*

DM : DN :: SX : SY;

T, D, S *shall be in a straight line.*

For because SX, SY are parallel to DM, DN, the angle MDN is equal to XSY. Also the sides are proportional by supposition: therefore the triangles XSY, MDN are similar; and the angle $SXY = DMN$. But the angle $SXT = DMT$; therefore $YXT = NMT$, and the triangles YXT, NMT are similar. Therefore

$$YX : XT :: NM : MT$$
$$\text{also } SX : YX :: DM : NM$$
$$\text{therefore } SX : XT :: DM : MT;$$

and the triangles SXT, DMT are similar, and the angle $DTM = STX$, whence D is in the straight line ST.

Prop. V. *Having given the velocity and direction of motion at three points of a figure described by a body acted on by a central force; to find the centre of force.*

Fig. 17. Let PT, QV, RZ be the three directions or tangents at P, Q, R, meeting in T and V. At P, Q, R erect three perpendiculars PA, QB, RC, reciprocally proportional to the velocities; that is, such that

$$PA : QB :: \text{vel. at } Q : \text{vel. at } P$$
$$QB : RC :: \text{vel. at } R : \text{vel. at } Q.$$

Draw AD, DBE, EC, parallel to PT, TV, VR, and meeting in D, E; join TD, VE; these lines produced to meet will give the centre of force.

For let S be the centre, SX, SY, and SZ perpendiculars upon the tangents: DM, DN parallel to SX, SY. By Prop. I. Cor. 1, we have

$SX : SY ::$ vel. at Q : vel. at $P :: PA : QB :: DM : DN$.

Hence, by the Lemma, SDT is a straight line. For a like reason SEV is a straight line. Therefore S is at the concourse of the two lines TD, VE.

Prop. VI. *If a body revolve about a fixed centre of force, and if, at the extremity of a small arc, the subtense of the angle of contact at any point be drawn parallel to the line joining the centre of force and that point;*

The force tending to the center is ultimately as the subtense directly, and as the square of the time of describing the arc inversely, when the time is indefinitely diminished.

Fig. 25. Let a body revolve in the line PQ, about the centre S: and PR being a tangent, let QR be parallel to SP.

Let F be the force at P, T the time of describing PQ. Then by Cor. 1 to the second Law of Motion, RQ is ultimately equal to the space described in T by F continued constant; that is, to $\frac{1}{2}FT^2$. Therefore

$$QR = \tfrac{1}{2}FT^2 \textit{ ultimately},$$

$$\text{and} \quad F = \frac{2QR}{T^2} \textit{ ultimately}.$$

Whence this proposition is manifest.

Cor. 1. Let QT be drawn perpendicular on SP: then (Lemma VIII) the area SPQ is *ultimately* equal to $\frac{1}{2}SP . QT$. But if A be the area described in a unit of time, by Prop. I.

Area $SPQ : A :: T : 1$; hence $\frac{1}{2}SP . QT = A . T$ *ultimately*,

$$\text{and} \quad \tfrac{1}{4}SP^2 . QT^2 = A^2 . T^2 \textit{ ultimately}.$$

Multiply this into the equation in the proposition, and we have $\frac{1}{4}SP^2 . QT^2 . F = 2A^2 . QR$ *ultimately*,

$$\text{whence} \quad F = \frac{8A^2 . QR}{SP^2 . QT^2} \textit{ ultimately}.$$

COR. 2. Let SY be drawn perpendicular on the tangent PR: then (Lemma VII and VIII) the area SPQ is *ultimately* equal to $\frac{1}{2} SY.PQ$: hence by the same reasoning as in the last corollary, we shall have

$$F = \frac{8A^2.QR}{SY^2.PQ^2} \text{ ultimately.}$$

COR. 3. Let PV be the chord of the circle of curvature passing through S: then, by Lemma XI, Cor. 1, $PV.QR = PQ^2$, and if we put $PV.QR$ for PQ^2 in Cor. 2, we find

$$F = \frac{8A^2}{SY^2.PV}.$$

COR. 4. If V be the velocity of the body at P, $PQ = V.T$ *ultimately* by Cor. 2 to the Definition of velocity; and $PQ^2 = V^2T^2$; therefore $PV.QR = V^2T^2$ *ultimately*: but $QR = \frac{1}{2}FT^2$ *ultimately*, as in the proposition: hence $\frac{1}{2}PV.F = V^2$, and

$$F = \frac{V^2}{\frac{1}{2}PV}.$$

COR. 5. If F be a constant force and V the velocity acquired in falling through a space S, $V^2 = 2FS$, by Cor. 4 to Lemma X. Now $V^2 = 2F.\frac{1}{4}PV$, by last Cor. Hence the velocity in any curve is equal to that acquired in falling down $\frac{1}{4}$ the chord of curvature passing through the center of force, the force being constant during this fall.

It appears from this Proposition and its corollaries, that a curve being given, which is described by the action of a central force, the law and magnitude of the force may be determined by determining the *ultimate* value of any of the following quantities:

$$\frac{2QR}{T^2}, \quad \frac{8A^2.QR}{SP^2.QT^2}, \quad \frac{8A^2.QR}{SY^2.PV^2};$$

or the value of either of the following quantities

$$\frac{8A^2}{SY^2.PV}, \quad \frac{V^2}{\frac{1}{2}PV}.$$

The determination of the law and magnitude of the force, when the curve is given, is the solution of *the direct Problem of central forces.*

The following Propositions are Examples of such solutions.

Prop. VII. *A body revolves in the circumference of a circle, the force tending to any point: to find the force* (F.)

Fig. 19. Let S be the centre of force, and SY perpendicular on the tangent PY. Then by Cor. 3, to last Prop.

$$F = \frac{8A^2}{SY^2 . PV}.$$

Let AV be the diameter: then by similar triangles AVP, PSY,

$$AV : PV :: SP : SY = \frac{PV . SP}{AV};$$

$$\text{putting this value for } SY,\ F = \frac{8A^2 . AV^2}{SP^2 . PV^3}.$$

Hence in different points of the curve the force varies inversely as $SP^2 . PV^3$.

Cor. 1. Hence if the centre of force be in the circumference of the circle, the points S and V coincide, and

$$F = \frac{8A^2 . AS^2}{SP^5}.$$

The force in this case varies inversely as the fifth power of the distance.

Cor. 2. Let the force $= \frac{\mu}{SP^5}$, μ being constant. Also let P be the periodic time: then $P . A =$ the whole area of the circle $= \frac{\pi . AS^2}{4}$. Hence $A = \frac{\pi . AS^2}{4P}$.

$$\text{And } \mu = (8A^2 . AS^2 =) \frac{\pi^2 . AS^6}{2P^2},$$

$$\text{whence } P^2 = \frac{\pi^2}{2\mu} AS^{4}, \; ^{6}$$

About the same centre, μ is the same, and $P \propto AS^{\frac{5}{2}}$.

Cor. 3. By Cor. 1. Prop. I.

$$\frac{\text{vel. at } P}{\text{vel. at } A} = \frac{SA}{SY} = \frac{SA}{SP} \cdot \frac{SP}{SY} = \frac{SA^2}{SP^2} \text{ by similar triangles.}$$

Cor. 4. Let R, S be two centres of force, A the area described in a unit of time, when the body moves in a circle about R; B the area described in a unit of time when the body moves in the same circle about S; F, G, the two forces.

$$\text{Then} \quad F = \frac{8A^2 . AV}{RP^2 . PT^3},$$

$$G = \frac{8B^2 . AV}{SP^2 . PV^3}.$$

Hence

$$F : G :: \frac{A^2}{RP^2 . PT^3} : \frac{B^2}{SP^2 . PV^3}, \text{ or } :: A^2 \frac{PV^3}{PT^3} : B^2 . \frac{RP^2}{SP^2}.$$

Let SG parallel to TP meet the tangent: then by similar triangles PSG, TPV, $PV : PT :: SG : SP$;

$$\text{hence } \frac{PV}{PT} = \frac{SG}{SP},$$

$$\text{And } F : G :: A^2 . \frac{SG^3}{SP^3} : B^2 \frac{RP^2}{SP^2}, \text{ or } :: A^2 . SG^3 : B^2 . SP . RP^2.$$

Cor. 5. In any curve touching GP at P, and having PTV for its circle of curvature, the subtenses of the angles of contact are *ultimately* equal to those of the corresponding arcs of the circle; and hence (Prop. VI) the forces in such a curve are ultimately equal to the forces in the circle in the same directions, and for the same velocity. Hence in any curve, if F be the force tending to R, G the force tending to S, SG parallel to RP, A and B the areas in a unit of time described about R and S respectively.

$$F : G :: A^2 . SG^3 :: B^2 . SP . RP^2.$$

Prop. VIII. Fig 20. *Let a body move in a semicircle* PQA, *the force tending in parallel lines* PM, QN: *to find the law and magnitude of the force.*

This may be considered as a case of a body acted on by a central force, by conceiving the centre to be indefinitely distant, so that all lines tending to it may be assumed to be parallel.

Let C be the centre, and CMN a diameter perpendicular to the direction of the force. On account of the similar triangles CPM, PZT, RZQ, we have $CP^2 : PM^2 :: PR^2 : QT^2$;

Also by the property of the circle $PR^2 = QR \cdot RW$, and *ultimately* $= QR \cdot PV = QR \cdot 2PM$. Hence

$CP^2 : PM^2 :: QR \cdot 2PM : QT^2$; $CP^2 \cdot QT^2 = QR \cdot 2PM^3$;

$$\frac{QR}{QT^2} = \frac{CP^2}{2PM^3} \text{ ultimately};$$

Also $\frac{A^2}{SP^2}$ is constant; for A is constant, and $SP = SC$ *ultimately*, when S is indefinitely distant, and is therefore constant.

$$\text{Now } \frac{8A^2}{SP^2} = \frac{A^2 \cdot 2MN^2}{\frac{1}{4}SP^2 \cdot QT^2} = \frac{2MN^2}{T^2},$$

because $\frac{1}{2} SP \cdot QT : A :: T : 1$, (Prop. I.)

Hence $\frac{MN}{T}$ is constant: let $\frac{MN}{T} = U$, and multiplying together the expressions for

$\frac{QR}{QT^2}$ and for $\frac{8A^2}{SP^2}$, we have the force at $P = \frac{CP^2 \cdot U^2}{PM^3}$.

Here U is the velocity parallel to CA.

Hence it appears that the force is inversely as PM^3.

Lemma to Prop. IX. *In the spiral which cuts all its rays at equal angles, rays at equal angular distances have everywhere equal ratios.*

Fig. 21. Let any angle PSQ be conceived to be divided into any number of equal parts, PSO, OSO', $O'SQ$; and let

pSq be an equal angle in another part of the curve, divided into an equal number of parts. *Ultimately*, when the number of parts in each case is indefinitely increased, the triangles will be all similar: for $SPO = SPR$ *ultimately*, and $Spo = Spr$; and the other angles of the triangles are equal by the construction: hence, whatever be the number of angles, we have such proportions as these,

$$\begin{aligned} SP &: SO :: Sp : So; \\ SO &: SO' :: So : So'; \\ SO' &: SQ :: So' : Sq; \text{ therefore} \\ SP &: SQ :: Sp : Sq. \end{aligned}$$

COR. 1. Let β be the angle which the curve makes with the ray: and let the angle QSP be $= \gamma$, and the ratio $SQ : SP$ be $1 : 1 + c$. Let the angle QSP be divided into equal parts, PSO, &c. each $= \frac{\gamma}{n}$; therefore the ratios $SO : SP$, &c. are all equal; and $SO^n : SP^n :: SQ : SP$; or

$$SO : SP :: SQ^{\frac{1}{n}} : SP^{\frac{1}{n}} :: 1 : (1 + c)^{\frac{1}{n}}.$$

Now if OM be an arc with centre S, and RN a perpendicular on SP, we have

$$RN = SO \cdot \sin PSO, \; NP = MP \textit{ ultimately}, = SP - SO;$$

$$\text{hence } \frac{RN}{NP} = \frac{SO \sin PSO}{SP - SO}, \text{ ultimately};$$

$$= \frac{\sin PSO}{\frac{SP}{SO} - 1}; \text{ whence}$$

$$\tan \beta = \frac{\sin \frac{\gamma}{n}}{(1 + c)^{\frac{1}{n}} - 1}, \; n \text{ being infinite};$$

$$= \frac{\gamma}{n(1 + c)^{\frac{1}{n}} - n}.$$

Now, by the binomial theorem,

$$n(1+c)^{\frac{1}{n}} - n = n\left\{1 + \frac{1}{n}c + \frac{\frac{1}{n}\cdot\left(\frac{1}{n}-1\right)}{1.2}c^2 + \&c.\right\} - n$$

$$= c + \frac{\frac{1}{n}-1}{1.2}c^2 + \frac{\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{1.2.3}c^3 + \&c.$$

and since n is infinite, this is $= c - \frac{c^2}{2} + \frac{c^3}{3} - \&c.$

$$\text{Hence } \tan\beta = \frac{\gamma}{c - \frac{c^2}{2} + \frac{c^3}{3} - \&c.} = \frac{\gamma}{\mathrm{l}(1+c)}$$

COR. 2. Hence $\mathrm{l}(1+c) = \frac{\gamma}{\tan\beta}$; and

$1 + c = e^{\frac{\gamma}{\tan\beta}}$, e being the base of natural logarithms.

COR. 3. If PSO, OSO', &c. be taken all equal, the similar triangles thus formed will be in geometrical progression for

$$\frac{OSO'}{PSO} = \frac{SO.SO'}{SP.SO} = \frac{SO^2}{SP^2}.$$

Hence the whole area will be the sum of this infinite decreasing geometrical progression *ultimately*, PSO being diminished without limit, (Lemma III) and this sum gives

$$\text{area} = \frac{\text{triangle } PSO}{1 - \frac{SO^2}{SP^2}}.$$

Now if the angle $PSO = \gamma$, and $\frac{SO}{SP} = \frac{1}{1+c}$,

$$\text{triangle } PSO = \tfrac{1}{2}\,SO.SP\sin\gamma,$$

hence

$$\text{area} = \frac{\frac{1}{2}SO.SP\sin\gamma}{1 - \frac{SO^2}{SP^2}} = \frac{1}{2}\,\frac{SP^2\frac{SP}{SO}\sin\gamma}{\frac{SP^2}{SO^2} - 1} = \frac{SP^2}{2}\,\frac{(1+c)\sin\gamma}{(1+c)^2 - 1}.$$

But, by Cor. 2, $\sin\gamma = \gamma$ *ultimately* $= (c - \frac{c^2}{2} + \frac{c^3}{3} - \&c.)\tan\beta$,

hence

$$\text{area} = \frac{SP^2}{2}\,\frac{(1+c)\,(c - \&c.)}{2c + c^2}\tan\beta = \frac{SP^2}{2}\,\frac{(1+c)\,(1-\&c.)}{2+c}\tan\beta,$$

and *ultimately*, since $\gamma = 0$ and $c = 0$, this area $= \dfrac{SP^2}{4}\tan\beta$.

Prop. IX. *Let the body revolve in the equiangular spiral just described: to find the force.*

Fig. 22. Let PSQ, pSq be two small equal angles; and since the angles SPR, Spr are also equal, and their supplements PRQ, prq; and also $SP : SQ :: Sp : Sq$ by the Lemma to this proposition; the figures $SPRQ$, $Sprq$ are similar. And QT will be a similar line in each figure; hence

$$\frac{QR}{QT} = \frac{qr}{qt} \text{ and } \frac{QR}{QT^2} : \frac{qr}{qt^2} :: \frac{1}{QT} : \frac{1}{qt} :: \frac{1}{SP} : \frac{1}{sp}.$$

Now if the angle PSQ be different from pSq, $\dfrac{QR}{QT^2}$ remains the same as before, by Lemma XI. Hence, in all cases,

$$\frac{QR}{QT^2} : \frac{qr}{qt^2} :: \frac{1}{SP} : \frac{1}{Sp};$$

$$\text{and } \frac{QR}{SP^2 \cdot QT^2} : \frac{qr}{Sp^2 \cdot qt^2} :: \frac{1}{SP^3} : \frac{1}{Sp^3}:$$

and the force is inversely as the cube of the distance.

Cor. 1. To find the actual value of the force: let

$$PSQ = \gamma,\ \frac{SP}{SQ} = 1 + c.$$

$$QR = TN = PT - PN = SP - SQ\cos PSQ - RN\,\text{cotan}\,RPN$$
$$= SQ\,(1 + c - \cos\gamma - \sin\gamma\cot\beta)$$

$$QT^2 = \sin^2 PSQ = SQ^2\sin^2\gamma, \text{ and } SQ = SP \text{ ultimately};$$

$$\text{hence } \frac{QR}{QT^2} = \frac{1 + c - \cos\gamma - \sin\gamma\cot\beta}{SP\sin^2\gamma}.$$

Now when γ is very small, by Cor. 2,

$$1 + c = e^{\frac{\gamma}{\tan\beta}} = 1 + \frac{\gamma}{\tan\beta} + \frac{\gamma^2}{1.2.\tan^2\beta} + \&c.$$

$$\cos\gamma = 1 - \frac{\gamma^2}{1.2} + \&c.$$

$$\sin\gamma\cot\beta = \frac{\gamma}{\tan\beta} + \&c.$$

$$\sin^2\gamma = \gamma^2 - \&c.$$

hence, neglecting terms beyond γ^2, because the ultimate value is wanted,

$$\frac{QR}{QT^2} = \frac{\frac{\gamma^2}{2\tan^2\beta} + \frac{\gamma^2}{2}}{SP.\gamma^2} = \frac{1}{2\,SP}\left(\frac{\cos^2\beta}{\sin^2\beta} + 1\right) = \frac{1}{2\,SP\sin^2\beta}.$$

$$\text{Hence force} = \frac{8A^2\,QR}{SP^2\,QT^2} = \frac{4A^2}{\sin^2\beta.SP^3}.$$

Cor. 2. The figure, taken to any distance from the centre, is always similar to itself. Hence (Lemma v) the area is as SP^2; and the area A being constant, the time of descending to the centre from P is as SP^2.

Cor. 3. By Cor. 3 of last Lemma, whole area $= \frac{SP^2}{4}.\tan\beta$.

$$\text{Hence time to centre} = \frac{SP^2}{4A}\tan\beta.$$

Let force $= \frac{\mu}{SP^3}$; hence by Cor. 1, $\mu = \frac{4A^2}{\sin^2\beta}$, $A = \frac{\mu^{\frac{1}{2}}\sin\beta}{2}$.

$$\text{Whence time to centre} = \frac{SP^2}{2\mu^{\frac{1}{2}}\cos\beta}.$$

Cor. 4. By Cor. 1 to Prop. I.

$$\frac{\text{vel. at } P}{\text{vel. at } p} = \frac{Sy}{SY} = \frac{Sp}{SP}, \text{ by similar triangles.}$$

Prop. X. *A body revolves in an ellipse, the centre of force being in the centre of the ellipse: to find the force.*

Fig. 23. Let CA, CB be the semiaxes major and minor, P any point, and DK a diameter conjugate to the diameter PG; QR parallel, QT perpendicular to CP; Qv parallel to the tangent PR; PF perpendicular to DK.

By Conics (Hustler, Ellipse, Prop. XVI.)

$$Pv \, . \, vG : Qv^2 :: CP^2 : CD^2.$$

also $\quad Qv^2 : QT^2 :: CP^2 : PF^2$ by similar triangles.

hence $\quad Pv \, . \, vG : QT^2 :: CP^4 : CD^2 \, . \, PF^2.$

$$:: CP^4 : AC^2 \, . \, BC^2, \text{ Hust. Prop. xv.}$$

$$\text{Therefore } \frac{Pv \, . \, vG}{QT^2} = \frac{CP^4}{AC^2 \, . \, BC^2}.$$

Write QR for its equal Pv; and for vQ, $2CP$, to which it is *ultimately* equal; and dividing,

$$\frac{QR}{CP^2 \, . \, QT^2} = \frac{CP}{2 AC^2 \, . \, BC^2};$$

whence we have (by Prop. VI. Cor. 3, SP being here CP.)

$$\text{Force} = \frac{4 A^2 \, . \, CP}{AC^2 \, . \, BC^2}.$$

Hence the force is as the distance CP.

Cor. Let the force $= \mu \, . \, CP$; therefore

$$\frac{4 A^2}{AC^2 \, . \, BC^2} = \mu, \; A = \mu^{\frac{1}{2}} \frac{AC \, . \, BC}{2},$$

$$\text{And the periodical time} = \frac{\text{whole area}}{A} = \frac{\pi \, . \, AC \, . \, BC}{A},$$

$$= \frac{2\pi}{\mu^{\frac{1}{2}}}.$$

Hence the periodic time is the same, about the same centre, whatever be the magnitude of the major axis of the ellipse.

PROP. *A*. *Having given, at a given point, the velocity and direction of the motion of a body, revolving about a given centre of force varying as the distance, to find the curve described.*

Fig. 24. Let PR be the given direction of projection, C being the centre. And let PV be four times the space through which a body must fall to acquire, by the action of the force at P, $(=\mu . CP)$ continued constant, the velocity which the body has at P. Hence PV is given by the formula for constant forces, $V^2 = 2F . \frac{PV}{4} = \frac{\mu . CP . PV}{2}$.

Draw DCK parallel to PR, PF perpendicular on it, and take in PF, PH so that $PH^2 = \frac{CP . PV}{2}$. Join CH; bisect it in I; join PI; take IM, IN each equal to IC or IH. Draw CM, CN; take $CA = PM$, $CB = PN$.

CA, CB are the positions and magnitudes of the semiaxes major and minor of the ellipse described.

By Conics (Hustler, Ellipse, XIX). If PV be the chord of curvature at P, and CD the conjugate semidiameter

$$PV = \frac{2CD^2}{CP}, \text{ whence } CD^2 = \frac{CP . PV}{2} = CH^2.$$

With centre I and radius IC describe a circle; it will pass through C, H, M, N, and F because CFH is a right angle. Therefore $PF . PH = PM . PN = CA . CB$.

$$\begin{aligned}
\text{Also } PC^2 + PH^2 &= PI^2 + IC^2 + PI^2 + IH^2, \\
&= PI^2 + IM^2 + PI^2 + IN^2, \\
&= (PI + IM)^2 + (PI - IN)^2, \\
&= PM^2 + PN^2, \\
&= CA^2 + CB^2.
\end{aligned}$$

Hence, since $CD = PH$, we have

$$PF \cdot CD = AC \cdot CB, \; PC^2 + CD^2 = AC^2 + CB^2.$$

And these are two equations which determine AC, BC. But these are two properties of the semiaxes major and minor. (Hust. Ell. XIV. and XV.) Therefore AC and BC have the due magnitude of the semiaxes of the ellipse, of which the chord of curvature at P is PV.

Also they have the due position. For let PH, CN meet in x, and CF, NH in y. The angles xFy, xNy being right angles, the circle on diameter xy will pass through F and N.

Also

angle $PNx = CNM = CFM = xFN$, because $CFx = MFN$.

Therefore PNx is equal to the angle in the alternate segment xFN, and therefore PN is a tangent to the circle xy at N.

Hence $PF \cdot Px = PN^2 = BC^2$. But (Hust. Ell. XI.) this is true if Cx be the position of the major axis. Therefore CA is the position of the major axis: and NCM is a right angle; therefore CB is the position of the minor axis.

Therefore the ellipse of which PV is the chord of curvature is rightly determined by the preceding construction. But the curve described by the body can be no other than this ellipse; for a body acted on by the force and having at P the velocity and direction which are here supposed, may move in an ellipse, of which the chord of curvature will be PV (Prop. VI. Cor. 5) and there cannot be two curves described by two bodies setting out with the same velocity, in the same direction from the same point, and acted on by the same force.

SECTION III.

ON THE MOTION OF BODIES IN CONIC SECTIONS, THE FORCE BEING IN THE FOCUS.

[NEWTON. PRINCIPIA. *Book I. Section III.*]

PROP. XI. *A body revolves in an ellipse; it is required to find the force tending to the focus of the ellipse.*

Fig. 26. Let S be the focus, PQ a small arc, QR the subtense parallel to SP, QT perpendicular to SP; C the centre, PC a diameter, Qv an ordinate parallel to the tangent at P, cutting SP in x.

Let CD, conjugate to CP, meet SP in E; PE is equal to the semiaxis major AC; for if H be the other forces, and HI parallel to CE, we have $SC : CH :: SE : EI$; therefore $EI = ES$ and $2PE = PE + ES + PE - EI = PS + PI = PS + PH$, because IH being parallel to the tangent at P, makes equal angles with PS, PH; (Hust. Ell. Prop. IV.) But $PS + PH = 2AC$, (Hust. Ell. Prop. I.) Hence $PE = AC$. Now we have

$$QR\ (= Px) : Pv :: PE : PC \text{ by similar triangles.}$$

$$\begin{array}{rcccccccc}
\text{Therefore} & Gv.QR & : & Gv.vP & :: & PE.PC & : & PC^2, \\
\text{by Conics,} & Gv.vP & : & Qv^2 & :: & PC^2 & : & CD^2, \\
\text{ultimately} & Qv^2 & : & Qx^2 & :: & 1 & : & 1; \\
\text{and by sim. tri.} & Qx^2 & : & QT^2 & :: & PE^2 & : & PF^2.
\end{array}$$

Compounding these four proportions, and observing that ultimately $Gv = 2PC$, we have *ultimately*,

$$2PC.QR : QT^2 :: PE^3.PC : CD^2.PF^2,$$
$$:: AC^3.PC : AC^2.BC^2,$$

$$\text{hence } \frac{2PC.QR}{QT^2} = \frac{AC.PC}{BC^2}, \ \frac{QR}{QT^2} = \frac{AC}{2BC^2}.$$

Now $\dfrac{2BC^2}{AC} = L$ the latus rectum (Hust. Ell. III.)

Hence $\dfrac{QR}{QT^2} = \dfrac{1}{L}$.

The force $= \dfrac{8 A^2 QR}{QT^2 . SP^2}$ is $= \dfrac{8 A^2}{L . SP^2}$,

Hence the force varies inversely as SP^2.

PROP. XII. *A body moves in a hyperbola: it is required to find the force tending to the focus.*

Fig. 27. As before let CA, CB be the semiaxes major and minor, PG, DK two conjugate diameters; S the focus, Qxv an ordinate, parallel to the tangent at P and meeting SP in x; QR the subtense parallel to SP.

Let SP meet CD in E; PE is equal to the semiaxis major; for if H be the other focus, and if, in the figure, HI were drawn, meeting SP produced in the point I, we should have $SC : CH :: SE : EI$. Therefore $EI = ES$, and $2 PE = EI + EP - (ES - EP) = IP - SP = HP - SP$, [because HI being parallel to the tangent at P, makes equal angles with PI, PH. (Hust. Hyp. Prop. IV.)] But $HP - SP = 2AC$. (Hyp. P. I.) Hence $PE = AC$. And we have

$$QR\ (= Px) : Pv :: PE : PC, \text{ by similar triangles.}$$

Therefore	$Gv.QR$	:	$Gv.vP$	::	$PE.PC$	: PC^2,
by Conics,	$Gv.vP$	:	Qv^2	::	PC^2	: CD^2,
ultimately,	Qv^2	:	Qx^2	::	1	: 1;
and by sim. tri.	Qx^2	:	QT^2	::	PE^2	: PF^2.

Compounding, we have *ultimately* (Gv being then $= 2PC$),

$$2PC.QR : QT^2 :: PE^2.PC : CD^2.PF^2,$$
$$:: AC^2.PC : AC^2.BC^2, \text{ by Conics.}$$

$$\text{Whence } \frac{QR}{QT^2} = \frac{AC}{2BC^2} = \frac{1}{L},$$

$$\text{Hence the force} = \frac{8A^2 QR}{QT^2.SP^2} = \frac{8A^2}{L.SP^2};$$

and the force varies inversely as the square of the distance.

In a similar manner it may be shewn that if the body move in the opposite hyperbola, it must be acted on by forces tending from, instead of to, the centre, and varying inversely as the square of the distance.

Prop. XIII. *Let a body move in a parabola: it is required to find the body tending to the focus of the figure.*

Fig. 28. Let S be the focus, A the vertex, P any point, QR the subtense parallel to SP, QT perpendicular to SP, PG parallel to the axis, Qxv parallel to the tangent; SN perpendicular to the tangent.

Then PG, PS make equal angles with the tangent, (Parab. II) to which Qxv is parallel: therefore xPv is an isosceles triangle; and $Pv = Px = QR$. Now by Conics (Parab. XI) $4SP.Pv = Qv^2$; or $4SP.QR = Qx^2$ ultimately (Lemma VI.)

But $Qx^2 : QT^2 :: SP^2 : SN^2 :: SP : SA$; (Parab. VIII.)

$$\text{whence } 4SP.QR : QT^2 :: SP : SA.$$

$$\text{And } \frac{QR}{QT^2} = \frac{1}{4SA},$$

$$\text{Hence the force} = \frac{8A^2.QR}{SP^2.QT^2} = \frac{2A^2}{SA.SP^2}.$$

Prop. XIV. *When several bodies revolve in ellipses about the same centre of force, varying inversely as the*

square of the distance, the principal latus rectum in each, is as the square of the area described by its ray in a unit of time.

By Prop. XI. The force in this case is $\frac{8A}{L \cdot SP^2}$, where A is the area described by a unit of time, and L the latus rectum. But about the same centre, the force is $\frac{\mu}{SP^2}$; therefore

$$\mu = \frac{8A^2}{L}, \quad L = \frac{8A^2}{\mu}, \text{ or } L \text{ is as } A^2.$$

Cor. Hence $A = \frac{\sqrt{\mu} \cdot \sqrt{L}}{2\sqrt{2}}$.

Prop. XV. *On the same suppositions, the periodic times in the ellipses, are in the sesquiplicate ratio of the major axis (that is in the ratio of the power of which the index is $\frac{3}{2}$.)*

For the whole area $= \pi \cdot AC \cdot BC$. Hence

$$\text{periodic time} = \frac{\text{whole area}}{\text{area in time 1}},$$

$$= \frac{\pi AC \cdot BC}{\frac{\sqrt{\mu}\sqrt{L}}{2\sqrt{2}}} = \frac{2\pi AC \cdot BC}{\sqrt{\mu}\frac{BC}{\sqrt{AC}}} = \frac{2\pi AC^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}.$$

Cor. Hence the periodic time depends only on the major axis, and is therefore the same in the ellipse, and in a circle of which the diameter is equal to the major axis of the ellipse.

Fig. 26. If PQ be an arc described in any time T, PQ is *ultimately* $= T \cdot V$, V being the velocity at P. Also the area SPQ is *ultimately* $= A \cdot T$. But $2SPQ$ is *ultimately* $= SY \cdot PQ$. Therefore $SY \cdot V \cdot T = A \cdot T$, and hence $V = \frac{A}{SY}$.

PROP. XVI. *On the same suppositions, the velocities of the bodies at any points, are inversely as the perpendiculars from the centre of force upon the tangents at these points, and directly as the square roots of the Latera recta of the orbits.*

$$\text{Now by Prop. XIV } A = \frac{\sqrt{\mu}\cdot\sqrt{L}}{2\sqrt{2}.};$$

$$\text{hence, by Prop. XV, } V = \frac{\sqrt{\mu}}{2\sqrt{2}}\cdot\frac{\sqrt{L}}{SY}$$

$$\text{and the velocity is as } \frac{\sqrt{L}}{SY}.$$

COR. 1. Hence $L = \frac{8}{\mu}.V^2.SY^2$, and $L \propto V^2.SY^2$.

COR. 2. At the greatest and least distances, the curve is perpendicular to the ray; SY coincides with the distance, (as SA,) and $V \propto \frac{\sqrt{L}}{SA}$.

COR. 3. To compare the velocity at any point of a conic section, with the velocity of a body revolving in a circle at the same distance, and acted on by the same force.

The velocity in any curve is equal to that acquired by falling down $\frac{1}{4}$ the chord of curvature, by the force, continued constant. Prop. VI, Cor. 3. Hence, PV being the chord of curvature,

$$V^2 \text{ in conic section} : V^2 \text{ in circle} :: 2F.\frac{PV}{4} : 2F.\frac{SP}{2}$$
$$:: PV : 2SP.$$

$$\text{Now in the ellipse or hyperbola } PV = \frac{2CD^2}{AC} = \frac{2SP.HP}{AC}.$$

$$\text{Hence } V^2 \text{ in conic section} : V^2 \text{ in curve} :: \frac{2.SP.HP}{AC} : 2SP.$$
$$:: HP : AC.$$

COR. 4. At the mean distance in the ellipses, $HP = AC$; hence the velocity is equal to that in the circle.

Cor. 5. In the same figure, or in different figures, for which the latus rectum is the same, $V \propto \frac{1}{SY}$.

Cor. 6. In the parabola $SY = \sqrt{SP \cdot SA}$, hence

$$V \propto \frac{1}{SY} \propto \frac{1}{\sqrt{SP \cdot SA}} \propto \frac{1}{\sqrt{SP}} \text{ because } SA \text{ is constant.}$$

In the ellipse $SY^2 = BC^2 \frac{SP}{HP}$ (Hust. Ell. VII) $= BC^2 \frac{SP}{2AC - SP}$

$$V \propto \frac{1}{SY} \propto \frac{\sqrt{2AC - SP}}{BC\sqrt{SP}}.$$

Hence, when SP diminishes, V increases in proportion to $\frac{1}{\sqrt{SP}}$, in consequence of the denominator: but it also increases in consequence of the numerator, for $2AC - SP$ increases; hence the velocity increases faster than in proportion to $\frac{1}{\sqrt{SP}}$.

In the hyperbola, $SY^2 = BC^2 \frac{SP}{HP} = BC^2 \frac{SP}{2AC + SP}$ (Hyp. VI.)

$$V \propto \frac{1}{SY} \propto \frac{\sqrt{2AC + SP}}{BC\sqrt{SP}}.$$

Hence when SP diminishes, V increases in proportion to $\frac{1}{\sqrt{SP}}$ in consequence of the denominator: but it diminishes in consequence of the numerator, for $2AC + SP$ diminishes: hence the velocity increases slower than in proportion to $\frac{1}{\sqrt{SP}}$.

Cor. 7. In the parabola $PV = 4SP$. Hence, by Cor. 3,

$$V^2 \text{ in parabola} : V^2 \text{ in circle} :: 2 : 1,$$

$$\text{and } V \text{ in parabola} : V \text{ in circle} :: \sqrt{2} : 1.$$

In the ellipse, $HP = 2AC - SP$; hence, by Cor. 3,

$$V \text{ in ellipse} : V \text{ in circle} :: \sqrt{2\,AC - SP} : \sqrt{AC}$$
$$:: \sqrt{2 - \frac{SP}{AC}} : 1.$$

Hence the ratio is less than $\sqrt{2} : 1$.

In the hyperbola $HP = 2\,AC + SP$; hence, by Cor. 3,

$$V \text{ in hyperbola} : V \text{ in circle} :: \sqrt{2\,AC + SP} : \sqrt{AC},$$
$$:: \sqrt{2 + \frac{SP}{AC}} : 1.$$

Hence the ratio is greater than $\sqrt{2} : 1$

Cor. 8. In the same way as in Cor. 3,

$$V \text{ in conic section} : V \text{ in circle of radius } \tfrac{1}{2}L :: \frac{\sqrt{L}}{SY} : \frac{\sqrt{L}}{\frac{1}{2}L}$$
$$:: \tfrac{1}{2}L : SY,$$

by Cor. 5, for the latus rectum L is the same for both curves.

Cor. 9. By the same reasoning

V in circle of radius $\frac{1}{2}L$: V in circle of radius $SP :: \sqrt{SP} : \sqrt{\frac{1}{2}L}$. Prop. VI. Cor. 3. Compounding this with the Proposition in last Cor.

$$V^2 \text{ in conic section at } P : V^2 \text{ in circle of radius } SP$$
$$:: \sqrt{SP \cdot \tfrac{1}{2}L} : SY.$$

Prop. XVII. *A given force being inversely as the square of the distance from the center, and the velocity and direction of the motion at a given point being known; to determine the curve described.*

Fig. 29. Let the force at P be $\frac{\mu}{SP^2}$, and V being the velocity at P, assume PV so that $V^2 = \frac{2\mu}{SP^2} \cdot \frac{PV}{4}$ or $PV = \frac{2\,V^2 SP^2}{\mu}$.

Draw PH making the angle $ZPH = YPS$.

If PV be less than $4\,SP$, take PH on the same side of YZ as S, and such that

$HP = \frac{PS \cdot PV}{4\,SP - PV}$, whence $PV = \frac{4SP \cdot HP}{SP + HP}$. If then we construct an ellipse with foci S and H, PV will be its chord of curvature through S; for, CD being conjugate to CP, chord of curvature $= \frac{2\,CD^2}{AC} = \frac{4\,CD^2}{2\,AC} = \frac{4\,SP \cdot HP}{SP + HP}$ by Conics.

If $PV = 4\,SP$, produce HP on the other side of YZ, take $PX = PS$, and draw a line WX perpendicular to PX. If we construct a parabola with focus S and directrix WP, the chord of curvature at P will be $4\,SP$ or PV.

If PV be greater than $4\,SP$, take Ph in HP produced, such that $hP = \frac{SP \cdot PV}{PV - 4\,SP}$, whence $PV = \frac{4\,SP \cdot hP}{hP - SP}$. If therefore we construct a hyperbola with foci S and h, PV will be its chord of curvature through S. For, CD being conjugate to CP, chord of curvature $= \frac{4\,C \cdot D^2}{2\,AC} = \frac{4\,SP \cdot hP}{hP - SP}$ by Conics.

And the conic section found by this construction will be the curve which the body describes: for if the body move in this conic section and have the velocity V at P, the force at P will be $\frac{\mu}{SP^2}$, by Prop. VI, Cor. 4. And there cannot be two curves which a body may describe, departing from P with the same velocity in the same direction, and acted on by a force of the same magnitude, and varying according to the same law: for if this were supposed, there would be nothing to determine which of the two curves the body should describe in any given instance. Therefore the conic section just found will be the path of the body.

Cor. 1. If the direction of motion at P be perpendicular to SP, P is the vertex of the curve, $PV = \frac{2\,CD^2}{AC} = \frac{2\,BC^2}{AC} = L$. And $HP = \frac{SP \cdot L}{4\,SP - L}$ in the ellipse; $hP = \frac{SP \cdot L}{L - 4\,SP}$ in the hyperbola.

COR. 2. If the velocity at the vertex be V, since at that point $SY = SP$, by Prop. XVI, $L = \frac{8}{\mu} V^2 . SP^2$; and the orbit may be constructed by the last corollary.

COR. 3. If the velocity of the body at any point be altered, the change in the orbit may be found by the proposition.

Thus at the point P of an ellipse, Fig. 29, let the velocity be altered in the ratio $1 : n$, the force remaining the same, since the direction of the motion is supposed to be unchanged, PH will be in the same direction as before. Let F be the force, V the velocity at P, then by the change V becomes nV: let PV become PV', and PH become PH'. Now

$$V^2 = 2F . \frac{PV}{4}, \quad n^2 V^2 = 2F \frac{PV'}{4}, \text{ whence } n^2 . PV = PV'.$$

$$\text{But } PV = \frac{4\,SP . HP}{SP + HP}, \text{ therefore } PV' = \frac{4\,n^2\,SP . HP}{SP + HP}.$$

$$\text{And } H'P = \frac{SP . PV'}{4\,SP - PV'} = \frac{4\,n^2 . SP^2 . HP}{4\,SP\,(SP + HP) - 4\,n^2\,SP . HP},$$

$$\frac{H'P}{HP} = \frac{n^2\,SP}{SP - (n^2 - 1)\,HP}.$$

And the position of H' being found, SH' the new position of the major axis is known.

$$\text{Also we have } H'P = \frac{n^2\,SP . HP}{SP - (n^2 - 1)\,HP},$$

$$SP + H'P = \frac{SP^2 + SP . HP}{n^2\,SP - (n^2 - 1)\,(SP + HP)}$$

(Since $SP + HP = 2\,AC$),

$$\frac{SP + H'P}{2\,AC} = \frac{SP}{n^2\,SP - (n^2 - 1)\,2\,AC}.$$

And $SP + H'P$ is the new axis major.

The centre C is the bisection of SH', and the new ellipse is determined.

COR. 4. If the force which acts on the body at any point be altered, the change in the orbit may be found by the proposition.

At the point P of an ellipse, Fig. 29, let the force be altered in the ratio 1 : m, the velocity remaining the same. As in last Cor. the direction of the motion being supposed to be unchanged, PH will continue the same direction as at first. Let F the force become mF, and PV, PH become PV', PH'. Then

$$V^2 = 2F.\frac{PV}{4},\quad V^2 = 2mF.\frac{PV'}{4},\ \text{whence } PV = m.PV'.$$

$$\text{But } PV = \frac{4SP.HP}{SP+HP},\ \text{therefore } PV' = \frac{4SP.HP}{m(SP+HP)}.$$

$$\text{And } H'P = \frac{SP.PV'}{4SP - PV'} = \frac{4SP^2.HP}{4mSP(SP+HP) - 4SP.HP},$$

$$\frac{H'P}{HP} = \frac{SP}{mSP - (1-m)HP}.$$

Hence the new position SH' of the major axis is found.

$$\text{Also we have } H'P = \frac{SP.HP}{mSP - (1-m)HP}$$

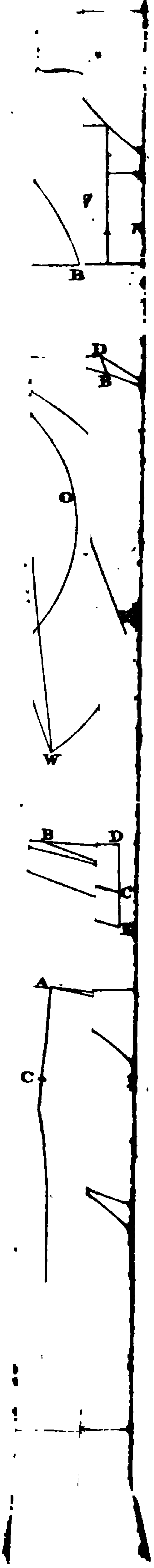
$$SP + H'P = \frac{mSP^2 + mSP.HP}{SP - (1-m)(SP+HP)}.$$

And since $SP + HP = 2AC$

$$\frac{SP + H'P}{2AC} = \frac{mSP}{SP - (1-m)2AC}.$$

And $SP + H'P$ is the new axis major.

The new centre C' is the bisection of SH' and the new ellipse is determined.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THE

FIRST THREE SECTIONS

OF

Sir Isaac, 1642-1727

NEWTON'S PRINCIPIA.

CAMBRIDGE:

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M.DCCC.XXXIV.

THE following pages have been taken with some slight alterations from the Manuscripts, which have been used of late years in St John's College, and are now printed with the view of saving to the Student the time and trouble, which it has hitherto been necessary to bestow in copying them. The few Propositions of the Seventh and Eighth Sections, now generally read in the University, will be found in the Appendix; and the Ninth and Eleventh Sections will be published, it is expected, in the course of a few weeks.

ST JOHN'S COLLEGE, *Jan.* 1834.

SECTION I.

OF THE METHOD OF LIMITS AND LIMITING RATIOS.

DEF. THE *Limit* of a continually increasing or decreasing quantity or ratio is that quantity or ratio, to which it continually approximates, but to which, though it may approach nearer than by any assignable difference, it never becomes actually equal.

OBS. The limit of a varying quantity or ratio is frequently called the *ultimate* value of that quantity or ratio; when we say that one quantity is *ultimately* equal to another, it is not to be inferred that the two quantities are ever equal, though their difference may be less than any assignable quantity.

LEMMA I.

Quantities and the ratios of quantities, which tend continually to equality, and whose difference may be made to bear to either of them a ratio less than any finite ratio, have their limits equal.

For if the limits be not equal, let L and $L + D$ represent them; then the difference of the limits to one of them

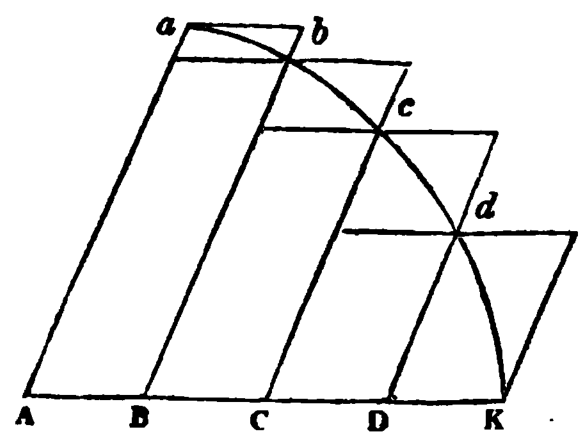
$$= D : L \text{ or } D : L + D.$$

Now since the quantities or ratios tend continually to equality, the ratio of their difference to either of them must always be greater than that of the difference of their limits to either of the limits, that is, than $D : L$ or $D : L + D$, either of which is a finite ratio. But by the hypothesis the ratio of their difference to either of them may be made less than any finite ratio, which is absurd; therefore the limits are not unequal, that is, they are equal.

COR. Hence if the quantities or ratios be finite, the limit of their difference, as they tend continually to equality, must equal 0. If they be indefinitely great, the limit of their difference may be a finite quantity or ratio, for it would bear to either of them an indefinitely small ratio. Lastly, if they be indefinitely small, it must be a quantity or ratio, which vanishes compared with either, that is, it must be a vanishing quantity or ratio of a higher order than either of them.

LEMMA II.

If in any figure AKa, bounded by the straight lines Aa, AK, and the curve line Ka, there be inscribed any number of parallelograms Ab, Bc, Cd,... on equal bases AB, BC, CD,..., and the parallelograms Ba, Cb, Dc.... be completed; then if the number of these parallelograms be increased and their breadths diminished indefinitely, the limit of the sum of each series will be the curvilinear area AKa.

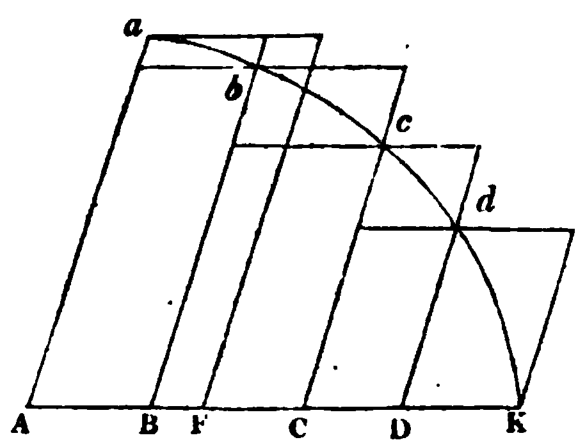


For as their bases are diminished, each series of parallelograms continually approximates to the area AKa. Also the difference between the two series is the sum of the parallelograms ab, bc, cd... which sum is equal to the parallelogram aB, for the base of each is equal to AB, and the sum of their altitudes to that of aB, and by diminishing the bases this difference, and therefore, *a fortiori*,

the difference of either series and the area AKa may be made less than any assignable quantity, and therefore, by Lemma I, the limit of either series is the curvilinear area AKa.

Lemma III.

If the two series of parallelograms be described in the same manner as in the last Lemma, except that their bases are not all equal, the limit of each series, when their bases are diminished indefinitely, is in this case also the curvilinear area AKa.



For take AF equal to the greatest base, and complete the parallelogram Fa; then this parallelogram, which is evidently greater than the difference between each series of parallelograms, may, by diminishing the base, be made less than any assignable quantity. Hence the difference between the two series, and therefore, *a fortiori*, the difference between each series and the area AKa, may be made less than any assignable quantity; and they tend continually to equality, therefore, by Lemma I, the limit of each series is the curvilinear area AKa.

Cor. 1. If the chords ab, bc, cd... be drawn, the limit of the area bounded by Aa, AK and the chords, when the bases AB, BC, CD.... are diminished indefinitely, is the curvilinear area AKa, for it always lies between this area, and the inner series of parallelograms.

Cor. 2. The limit of the figure bounded by Aa, AK and the tangents through a, b, c... is the same curvilinear area, since it lies always between the curvilinear area, and the outer series of parallelograms.

Cor. 3. The curve line aK is the limit of the boundary formed by the chords.

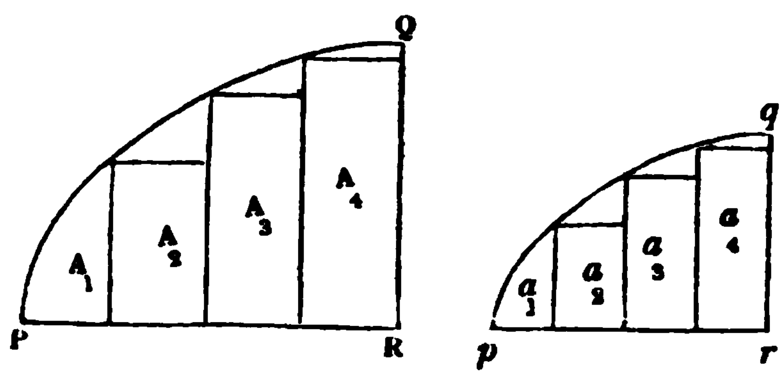
Lemma IV.

If in two curvilinear figures there can be inscribed the same number of parallelograms, which, when their number is increased and their breadths diminished indefinitely, are ultimately to each other in a given ratio, the areas of the curvilinear figures will be in that ratio.

Let PQR, pqr be the figures, and let the parallelograms $A_1, A_2, A_3, \ldots$ be inscribed in the one, and $a_1, a_2, a_3, \ldots$ in the other,

and let $\dfrac{A_1}{a_1} = m + x_1$, $\dfrac{A_2}{a_2} = m + x_2$, $\dfrac{A_3}{a_3} = m + x_3$, &c. = &c.

$x_1\ x_2\ x_3 \ldots$ being quantities, which vanish, when the breadths of the parallelograms are diminished indefinitely, so that according to the hypothesis,



$$\text{limit}\ \frac{A_1}{a_1} = m = \text{limit}\ \frac{A_2}{a_2} = \text{limit}\ \frac{A_3}{a_3} = \&\text{c.}$$

Hence $A_1 = m a_1 + x_1 a_1$, $A_2 = m a_2 + x_2 a_2$, $A_3 = m a_3 + x_3 a_3$, &c. = &c.

$$\therefore\ A_1 + A_2 + A_3 + \ldots = m(a_1 + a_2 + a_3 + \ldots) + x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots$$

$$\therefore\ \frac{A_1 + A_2 + A_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = m + \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots}{a_1 + a_2 + a_3 + \ldots}$$

$$\therefore\ \text{limit}\ \frac{A_1 + A_2 + A_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = m + \text{limit}\ \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots}{a_1 + a_2 + a_3 + \ldots}$$

Now since x_1 vanishes in the limit, $x_1 a_1$ is a vanishing quantity of a higher order than a_1; similarly $x_2 a_2$ vanishes compared with a_2, $x_3 a_3$ compared with a_3, and so on; the number of terms also in the two series is the same, therefore ultimately $x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots$ vanishes compared with $a_1 + a_2 + a_3 + \ldots$,

$$\text{or limit}\ \frac{x_1 a_1 + x_2 a_2 + x_3 a_3 + \ldots}{a_1 + a_2 + a_3 + \ldots} = 0.$$

$$\text{Also limit } \frac{A_1 + A_2 + A_3 + \dots}{a_1 + a_2 + a_3 + \dots} = \frac{\text{area } PQR}{\text{area } pqr},$$

$$\therefore \frac{\text{area } PQR}{\text{area } pqr} = m.$$

Cor. If there be two quantities of any kind, which are divided into the same number of parts, if these parts, when their number is continually increased and the magnitude of each continually diminished, are to each other in a given ratio, the whole quantities will be in that ratio.

For if the parts be substituted for the parallelograms, and the whole quantities for the figures PQR, pqr, the reasoning will be the same in the two cases.

Def. 1. A *curve* is a line traced out by a moving point, which is continually changing the direction of its motion.

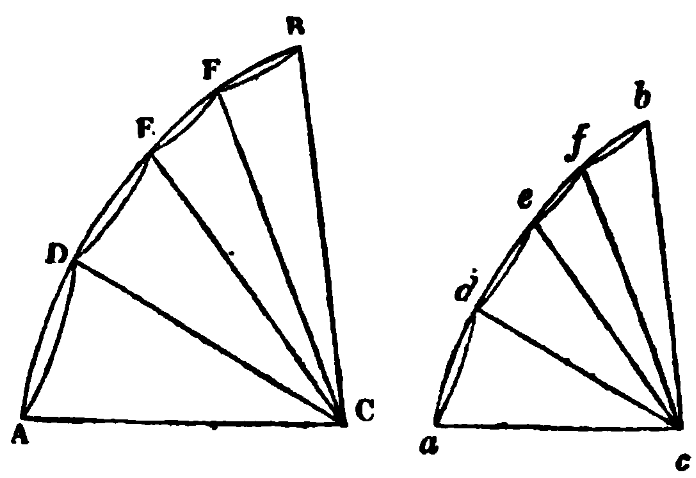
Def. 2. One curvilinear figure is said to be similar to another, when any rectilinear figure being inscribed in the first, a similar rectilinear figure may be inscribed in the other.

Obs. The curves and curvilinear figures, treated of in this Section, are always supposed to lie in one plane.

Lemma V.

The homologous sides of all similar curvilinear figures are proportionals, and their areas are in the duplicate ratio of the sides.

Let ACB, acb be two similar figures, of which the sides AB, AC, BC, are homologous to ab, ac, bc, respectively; then by definition, if $ADEBC$ be a polygon inscribed in ABC, a similar polygon $adebc$ may be inscribed in abc. Join CD, CE, and cd, ce, &c., dividing the polygons into the same number of similar triangles,



$$\therefore\ AD : AC = ad : ac,$$

$$\text{alt}^{\text{do}}\ AD : ad = AC : ac,$$

$$\text{Similarly}\ DE : de = DC : dc = AC : ac,$$

$$EF : ef = AC : ac,$$

....

therefore, componendo

$$AD + DE + EF + \&\text{c.} : ad + de + ef + \ldots = AC : ac.$$

Now this being always true, will be true when the number of sides is increased, and their magnitudes diminished, without limit;

$$\therefore \text{limit}\ AD + DE + EF + \ldots : \text{limit}\ ad + de + ef + \ldots = AC : ac,$$

and therefore by Lem. III, Cor. 3.

$$ADB : adb = AC : ac$$
$$= BC : bc.$$

Again, polygon $ADEBC$: polygon $adebc = AC^2 : ac^2$,

and this being always true will be true in the limit;

$$\therefore \text{limit polygon}\ ADEBC : \text{limit}\ adebc = AC^2 : ac^2;$$

therefore by Lem. III, Cor. 1,

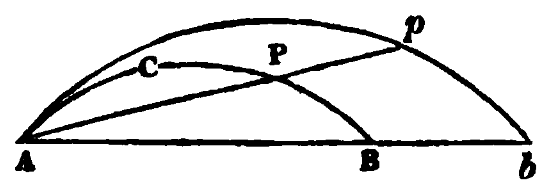
curvilinear figure ABC : curvilinear fig. $abc = AC^2 : ac^2$

$$= \overline{ADB}|^2 : \overline{adb}|^2$$
$$= BC^2 : bc^2.$$

Cor. If ACB, acb be two similar figures, and CE, ce be equally inclined to AC, ac, then $AC : CE = ac : ce$. Hence also this definition,

Two curves are said to be similar, when there can be drawn in them two distances from two points similarly situated, such, that if any two other distances be drawn equally inclined to the former, the four are proportional.

PROB. Let the chord AB of the curve ACB be produced to b, to describe on Ab a curve similar to ACB.



In ACB take any point P, join AP, and produce AP to p, so that $Ap : Ab = AP : AB$; then if the curve Apb be the locus of all points, whose position is determined in the same manner as that of p, it will be similar to the curve APB.

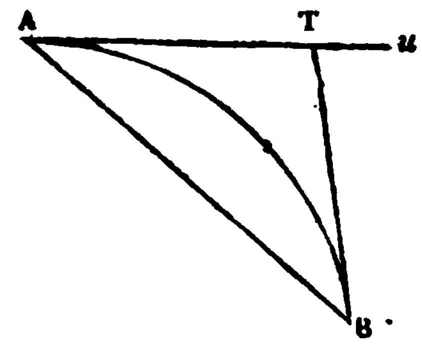
DEF. 1. The *tangent* to a curve AB at A is the straight line, in which the generating point would move, if instead of changing the direction of its motion, it moved on in the direction, which it had at A.

DEF. 2. The curvature of a curve is said to be *continued*, when the curve is wholly convex or concave to a given straight line on the same side of it, and when the change of direction is not abrupt, but gradual; that is, if ATU, BT, (Fig. Lem. VI.) be tangents at A and B, in a curve of continued curvature, the angle BTU as B moves up to A, diminishes through every change of magnitude from its original value and ultimately vanishes.

LEMMA VI.

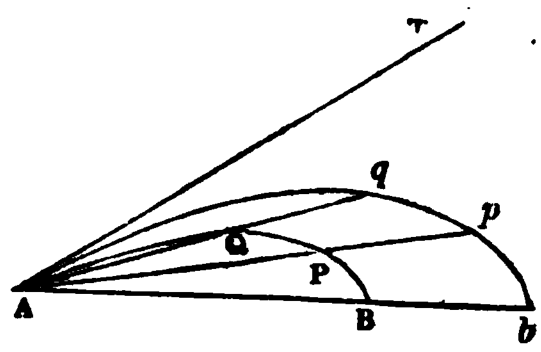
If ACB be an arc of continued curvature, AB the chord, and ATU the tangent at A, the angle BAT between the chord and tangent, as B moves along the curve towards A, and ultimately coincides with that point, continually diminishes and ultimately vanishes.

Let the tangents at A and B meet in the point T; then the angle BTU measures the change in the direction of the



motion of the generating point which takes place in passing from B to A, and since the curvature is continued, this angle, as B moves towards and ultimately coincides with A, continually diminishes and ultimately vanishes, therefore *a fortiori* the interior angle BAT continually diminishes and ultimately vanishes.

COR. Similar conterminous arcs, which have their chords coincident, have a common tangent.

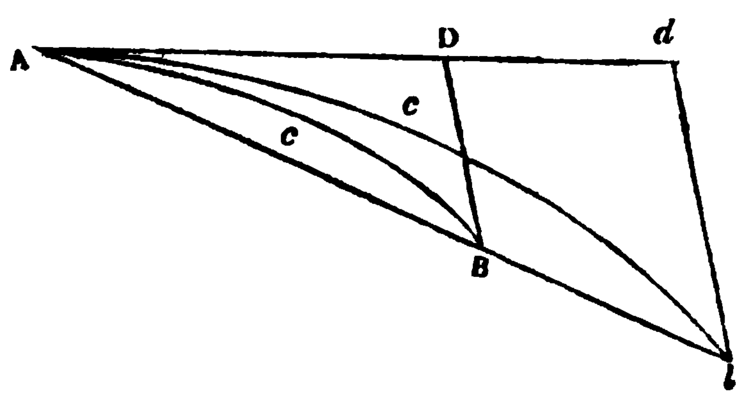


Let the similar conterminous arcs APB, apb have their chords AB, Ab coincident, and let APp, AQq be any other coincident chords; then since the curves are similar $AP : Ap = AB : Ab = AQ : Aq$, therefore the arcs AP, Ap are similar, that is, the chords of the similar arcs AP, Ap coincide. Now let P and p move up to A, the arcs AP, Ap, since they are always similar, will vanish together, and APp in its ultimate position will be a tangent to each, that is, the arcs AB, Ab have a common tangent.

DEF. The *subtense* of an arc is a straight line, drawn from one extremity of the arc to meet at a finite angle the tangent to the arc at its other extremity.

LEMMA VII.

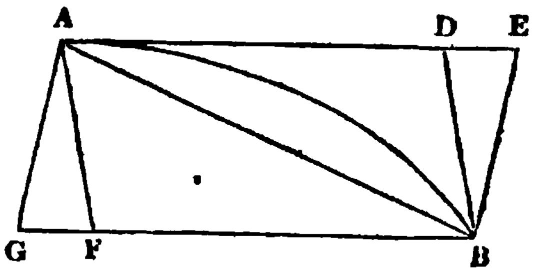
If BD be a subtense of the arc ACB of continued curvature, the chord AB, the arc ACB, and the tangent AD, when BD moves parallel to itself up to A, are ultimately equal to each other.



Produce AD to any fixed point d, and draw db parallel to DB to meet AB produced in b; on Ab describe the arc Acb similar to ACB, and as B moves up to A, let Acb so alter its form as to be always

similar to ACB; hence the two arcs have a common tangent, and the three lines AB, ACB, AD are always proportional to Ab, Acb, Ad. Now as B moves up to A, the angle bAd continually diminishes and ultimately vanishes, (Lemma VI.), the point b moves up to and coincides with d, and therefore Ab and Ad, and therefore, *a fortiori*, the intermediate arc Acb, are ultimately equal. Hence AB, ACB, AD, which are always proportional to them, are ultimately equal to each other.

COR. 1. Since the proof holds whatever be the inclination of BD to the tangent, provided it be finite, if BE be a subtense making any other finite angle with AD, the tangents AE, AD, and the chord and are are ultimately equal.

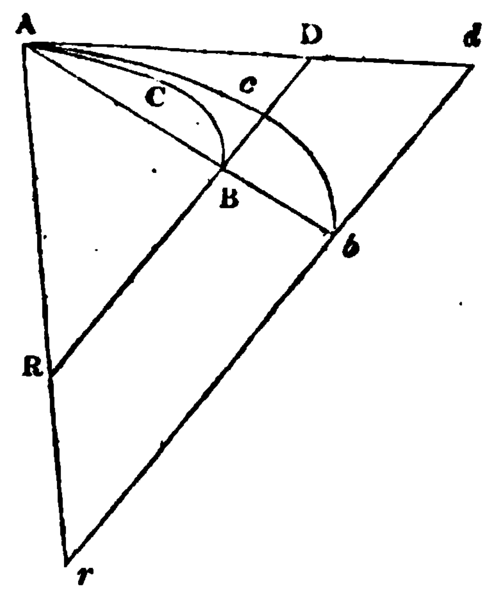


COR. 2. Also if the parallelograms $ADBF$, $AEBG$ be completed, since AD, AE are always equal to BF, BG respectively, the lines AD, AE, BF, BG are ultimately equal to the chord and arc; and in all geometrical investigations the ultimate values of all these lines may be used indiscriminately for each other.

LEMMA VIII.

If the straight lines AR, DBR, which meet in R, make with the chord AB, the arc ACB, and the tangent AD, the triangles ABR, $ACBR$, ADR; these three triangles when B moves up to A, are ultimately similar and equal to each other.

Produce AD to a fixed point d, and draw dbr parallel to DBR, meeting AB, AR produced in b, r. On Ab describe the arc Acb similar to ACB, and let it so alter its form, as B moves up to A, as to be always similar to ACB. Then the two arcs will have a common tangent ADd, and the three triangles ABR, $ACBR$, ADR will be always similar to the three Abr, $Acbr$, Adr respectively, and will bear each to each the same ratio, viz. that of $RA^2 : rA^2$; hence, alternando, $ABR : ACBR : ADR = Abr : Acbr : Adr$.

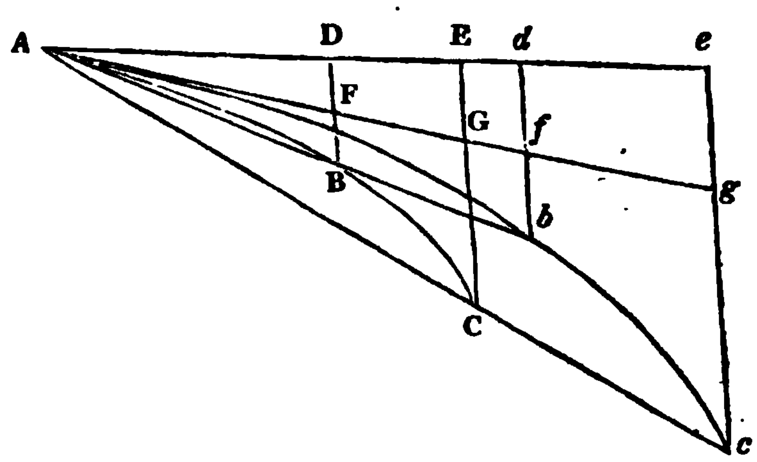


Now let BD move parallel to itself up to A, then the angle bAd continually diminishes and ultimately vanishes; and Ab and therefore the intermediate arc Acb ultimately coincide with Ad; hence the triangles Abr, $Acbr$, are ultimately similar and equal to Adr; therefore the triangles ABR, $ACBR$, ADR, which are always proportional to them, are ultimately similar and equal to each other.

Obs. In the Lemma RBD is supposed to move parallel to itself towards A, that is, b moves along rd fixed, and the triangles Abr, $Acbr$, Adr are always finite; but the same thing will be true, if RBD revolve round R fixed, in which case also, though r moves off to an infinite distance, and the triangles Abr, $Acbr$, Adr increase indefinitely, they will be ultimately similar and equal to each other.

Lemma IX.

If the right line AE and the arc ABC, given in position, cut each other in a finite angle at A, and the ordinates BD, CE be drawn, making any other given angle with AE; when BD, CE move parallel to themselves up to A, the limiting ratio of area ABD : area ACE equals that of $AD^2 : AE^2$.



Produce AE to a fixed point e, and take Ad in Ae such, that $Ad : Ae = AD : AE$. Draw db, ec parallel to DB, or EC, meeting AB, AC produced in b, c; and on Ac describe an arc similar to ABC : this arc shall pass through b, for by similar triangles and by construction,

$$AB : Ab = AD : Ad = AE : Ae = AC : Ac,$$

and therefore (Cor. Lemma v.) b is a point in the arc. As B and C move up to A, let the curve Abc so alter its form as to

be always similar to ABC, then the area ABD will be always similar to Abd, and ACE to Ace. Hence

$$\text{area } ABD : \text{area } Abd = AD^2 : Ad^2 = AE^2 : Ae^2$$
$$= \text{area } ACE : \text{area } Ace,$$
$$\therefore \text{ area } ABD : \text{area } ACE = \text{area } Abd : \text{area } Ace.$$

Also the two arcs being similar have a common tangent at A, let this be $AFGfg$; and let BD, CE move parallel to themselves up to A; then the angle cAg continually diminishes and ultimately vanishes, and therefore

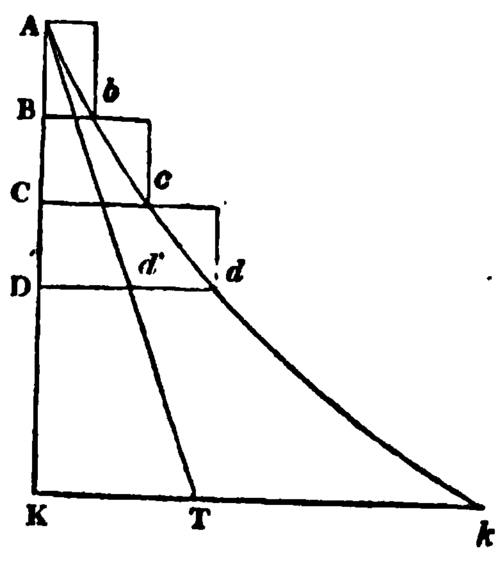
$$\text{L. R.* area } Abd : \text{area } Ace = \text{L. R. } \triangle Afd : \triangle Age$$
$$= \text{L. R. } Ad^2 : Ae^2$$

$$\text{Hence L. R. area } ABD : \text{area } ACE = \text{L. R. area } Abd : \text{area } Ace$$
$$= \text{L. R. } Ad^2 : Ae^2$$
$$= \text{L. R. } AD^2 : AE^2.$$

Lemma X.

The spaces, described from rest by a body acted on by any finite force, are in the beginning of the motion as the squares of the times, in which they are described.

Def. A finite accelerating or retarding force is such, that the ratio of the time to the velocity generated or destroyed in that time is finite.



Let the straight line AK represent the time of the body's motion from rest, and Kk, drawn at right angles to AK, the last acquired velocity; suppose the time divided into equal intervals AB, BC, CD &c., and let Bb, Cc, Dd &c., drawn at right angles to AK, represent the velocities acquired in the times AB, AC, AD &c.; let $Abcdk$ be the curve passing through the extremities of all the ordinates thus drawn; and complete the parallelograms Ab, Bc, Cd &c.

* L. R. signifies "limit of the ratio" or "limiting ratio."

If now the force be supposed to act by impulses, which would cause the body to move uniformly during the times AB, BC, CD &c., with the velocities Bb, Cc, Dd &c. respectively, the spaces described in the 1st, 2d, 3d &c. intervals will be represented by the parallelograms Ab, Bc, Cd &c. On this supposition therefore, the space described in time AD : space in time AK = sum of the parallelograms in the former case : sum in the latter; and this being true always, will be true when the intervals are diminished and their number increased indefinitely, in which case the force, which was supposed to act by impulses, approximates to a continued force, and the sums of the parallelograms to the areas ADd, AKk.

Hence

space in time AD : space in time AK = area ADd : area AKk.

Let the tangent at A cut Kk in T; now, the force being finite, the ratio $AK : Kk$ is always finite; $\therefore$ $AK : KT$, which equals L. R. $AK : Kk$ is a finite ratio, and therefore,

$$\tan KAT \left(= \frac{KT}{KA}\right) \text{ is finite,}$$

or KA makes a finite angle with the curve at A;

Hence by Lemma IX,

$$\text{L. R. area } ADd : \text{area } AKk = \text{L. R. } AD^2 : AK^2,$$

and therefore in the beginning of the motion, space $\propto$ (time)^2.

Cor. 1. Force is measured by the velocity generated in any time, divided by the time, the force being supposed to remain constant for that time. Hence if Dd' be the velocity generated by the force at A, continued constant, in time AD,

$$F \text{ at } A = \frac{Dd'}{AD},$$

and this being always true, will be true when AD is diminished indefinitely,

$$\therefore\ F = \text{limit}\,\frac{Dd'}{AD} = \text{limit}\,\frac{Dd}{AD}$$

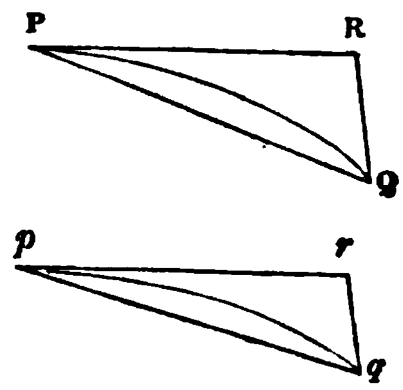
$$= \frac{KT}{AK} = \frac{KT\,.\,AK}{AK^2} = \frac{2\text{ triangle }AKT}{AK^2} = 2\text{ limit}\,\frac{\text{area }AKk}{AK^2}$$

$$= 2\text{ limit}\,\frac{\text{space}}{(\text{time})^2}.$$

Cor. 2. The effect produced by F upon the body is independent of any motion which it may have, when F begins to act upon it. Hence generally if S be the space, through which a force F, acting on a body moving in any orbit, draws the body in T'' from the place it would have occupied if the extraneous force had not acted, $F = 2\text{ limit}\,\frac{S}{T^2}$.

On the Curvature of Curve Lines.

Prop. I. If in PR, pr tangents at the points P, p in the curves PQ, pq, PR be taken equal to pr, and the subtenses QR, qr be drawn equally inclined to them, then when QR, qr move parallel to themselves to P, p,



$$\frac{\text{curvature of }PQ\text{ at }P}{\text{curvature of }pq\text{ at }p} = \text{limit}\,\frac{QR}{qr}.$$

Draw the chords PQ, pq,

$$\text{then}\ \frac{\text{curvature of }PQ\text{ at }P}{\text{curvature of }pq\text{ at }p} = \frac{\text{angle of contact at }P}{\text{angle of contact at }p}$$

$$= \text{limit}\,\frac{\text{angle }QPR}{\text{angle }qpr}$$

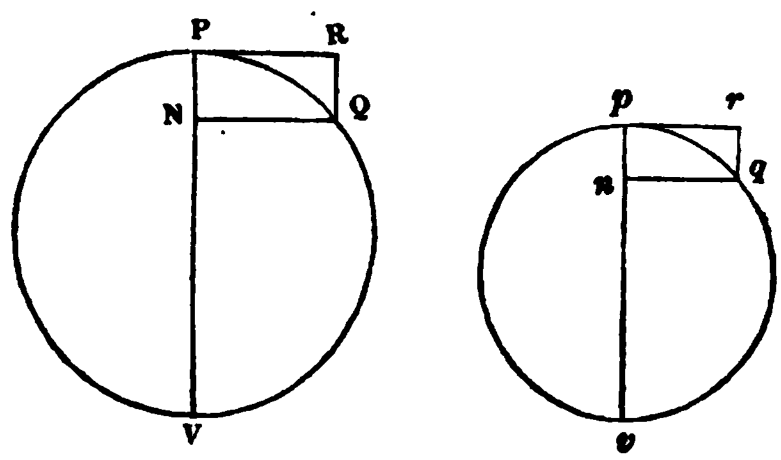
$$= \text{limit} \frac{\sin QPR}{\sin qpr}$$

$$= \text{limit} \frac{\frac{QR}{RP} \sin R}{\frac{qr}{rp} \sin r}$$

$$= \text{limit} \frac{QR}{qr}.$$

PROP. II. The curvatures in different circles vary inversely as the diameters.

Let PQV, pqv be two circles, draw the diameters PV, pv, and the tangents PR, pr. Take $PR = pr$, and draw the subtenses QR, qr parallel to the diameters, and QN, qn parallel to the tangents;



$$\text{then } \frac{QR}{qr} = \frac{PN}{pn} = \frac{QN^2}{NV} \div \frac{qn^2}{nv} = \frac{nv}{NV},$$

$$\therefore \frac{\text{curvature at } P}{\text{curvature at } p} = \text{limit} \frac{QR}{qr}$$

$$= \text{limit} \frac{nv}{NV}$$

$$= \frac{pv}{PV}$$

$$\text{or the curvature} \propto \frac{1}{\text{diameter}}.$$

COR. Hence in the same circle the curvature is the same at every point.

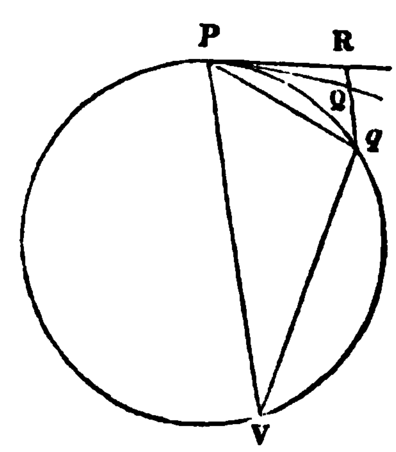
From this property of the circle, and also because by varying the diameter it may be made to have any curvature we please, the circle is made use of to measure the curvature at any proposed points of other curves.

DEF. The *circle of curvature* at any point of a curve is that circle, which has the same tangent and curvature as the curve has at that point.

Hence if QqR be a common subtense to the curve PQ and the circle Pq, and limit $\dfrac{QR}{qR} = 1$, Pq will be the circle of curvature at P.

The radius, diameter and chord of the circle of curvature are generally called the radius, diameter, and chord of curvature.

PROP. III. If PqV be the circle of curvature at any point P, and PV a chord drawn in any given direction, then



$$PV = \text{limit} \frac{(\text{arc})^2}{\text{subtense parallel to the chord}}.$$

Take PQ a small arc of the curve, through Q draw the subtense RQq parallel to PV, and join Pq, qV; then since the triangles PRq, PqV are evidently similar,

$$PV = \frac{Pq^2}{qR}$$

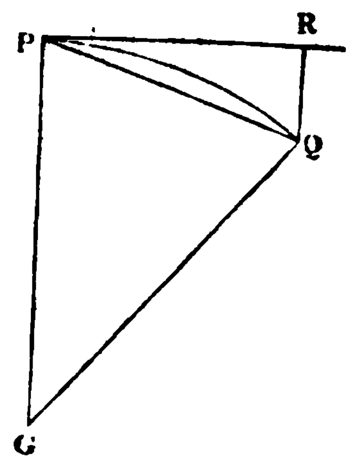
Now this being true whatever be the magnitude of PQ, will be true, when RQr moves parallel to itself up to P, in which case $Pq = PQ$ ultimately, and $qR = QR$ ultimately,

$$\therefore\ PV = \text{limit}\,\frac{Pq^2}{qR}$$

$$= \text{limit}\,\frac{(\text{arc } PQ)^2}{QR}.$$

Cor. Hence the diameter of curvature

$$= \text{limit}\,\frac{(\text{arc})^2}{\text{subtense perpendicular to the tangent}}.$$



Prop. IV. If in the curve PQ, PG and QG, drawn perpendicular to the tangent PR and the chord PQ respectively, intersect in G, then when Q moves up to P, the limit of PG is the diameter of curvature at P.

Draw the perpendicular subtense QR,

Then by similar triangles PQR, PGQ

$$PG = \frac{PQ^2}{QR};$$

$$\therefore\ \text{limit } PG = \text{limit}\,\frac{PQ^2}{QR}$$

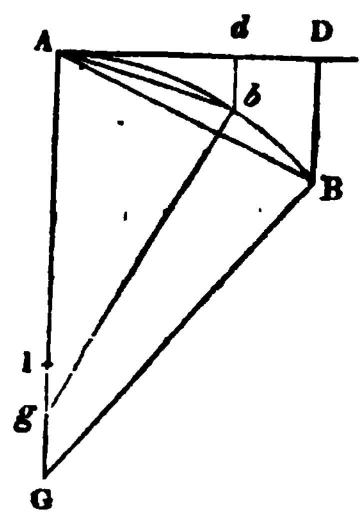
$$= \text{limit}\,\frac{(\text{arc } PQ)^2}{QR}$$

$$= \text{diameter of curvature at } P.$$

Def. The curvature of a curve at any point is said to be finite, when the diameter of curvature at that point is finite.

Lemma XI.

In curves of finite curvature the limiting ratio of the subtenses equals that of the squares of the conterminous arcs.



Let AbB be the curve having a finite curvature at A;

First, Let the subtenses bd, BD be perpendicular to the tangent at A. Draw bg, BG at right angles to the chords Ab, AB, and let them meet AgG, which is drawn at right angles to the tangent AD, in the points g and G.

Then as b and B move up to A, g and G move up to I, the extremity of the diameter of curvature of A, as their limit. (Prop. IV.)

Now by similar triangles,

$$BD = \frac{AB^2}{AG}, \quad bd = \frac{Ab^2}{Ag},$$

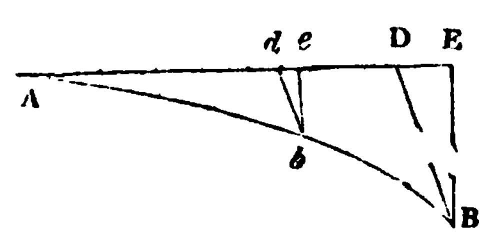
$$\therefore BD : bd = \frac{AB^2}{AG} : \frac{Ab^2}{Ag},$$

$$\therefore \text{L. R. } BD : bd = \text{L. R. } \frac{AB^2}{AG} : \frac{Ab^2}{Ag}$$

$$= \text{L. R. } AB^2 : Ab^2,$$

(since AG, Ag are ultimately equal to AI)

$$= \text{L. R. } (\text{arc } AB)^2 : (\text{arc } Ab)^2.$$



Secondly, Let the subtenses be inclined at any equal angles to the tangent. Draw BE, be perpendicular to the tangent: then by similar triangles,

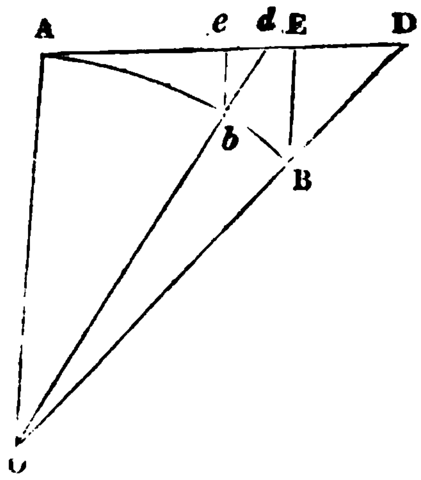
$$BD : BE = bd : be,$$

alternando $BD : bd = BE : be$;

$$\therefore \text{ L. R. } BD : bd = \text{L. R. } BE : be$$

$$= \text{L. R. } (\text{arc } AB)^2 : (\text{arc } Ab)^2.$$

Thirdly, Let the subtenses, inclined at unequal angles to the tangent, converge to a point, and revolve round that point fixed, or approach to A according to any other given law.



Let O be the point in which DB, db meet when produced; draw BE, be always parallel to AO; then since the angles at D and d are always finite, AO must always be finite, and L. R. $DO : AO$ will be a ratio of equality, as also L. R. $do : AO$.

$$\left.\begin{array}{l}\text{But } BD : BE = DO : AO \\ \text{and } bd : be = do : AO\end{array}\right\}, \text{ always and therefore ultimately;}$$

$$\therefore \text{ L. R. } BD : BE = \text{L. R. } bd : be;$$

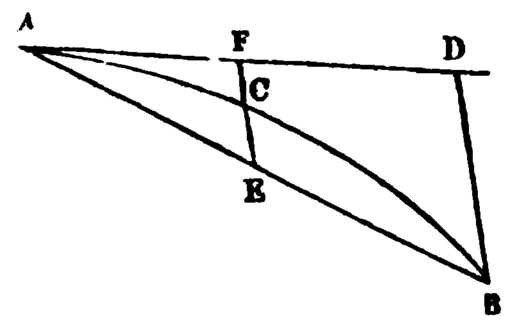
$$\therefore \text{ L. R. } BD : bd = \text{L. R. } BE : be$$

$$= \text{L. R. } (\text{arc } AB)^2 : (\text{arc } Ab)^2.$$

Cor. 1. Hence by Lemma vii. the limiting ratio of the subtenses will equal that of the squares of the arcs, chords, and tangents.

Theorem. *The subtense of an arc is ultimately equal to four times the parallel sagitta.*

Def. The sagitta of an arc is a line drawn at a finite angle to the chord from its middle point to meet the arc.



Let BD be a subtense of the arc AB, EC the sagitta parallel to it, bisecting the chord in E, and produced to meet the tangent in F.

Then by similar triangles,

$$AF = \tfrac{1}{2}AD, \text{ and } EF = \tfrac{1}{2}BD.$$

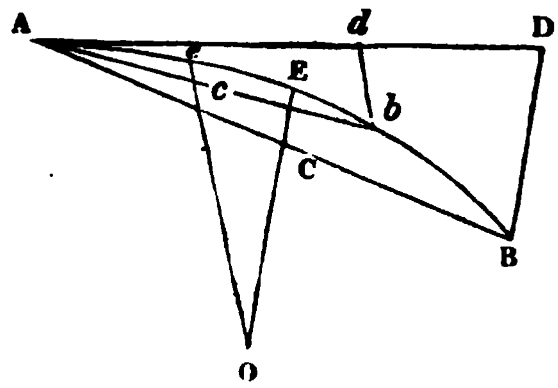
Also by the Lemma,

$$\text{L. R. } CF : BD = \text{L. R. } AF^2 : AD^2$$

$$= 1 : 4$$

$$\therefore \text{ L. R. } CE : BD = 1 : 4.$$

COR. 2. The limiting ratio of the sagittæ, which bisect the chords and converge to a given point, equals that of the squares of the arcs, chords, and tangents.



Let EC, ec be the sagittæ of the arcs AEB, Aeb, bisecting the chords AB, Ab in C, c; draw the subtenses BD, bd respectively parallel to them;

then $\text{L. R. } EC : BD = 1 : 4$

$$= \text{L. R. } ec : bd;$$

$$\therefore \text{ L. R. } EC : ec = \text{L. R. } BD : bd;$$

$$= \text{L. R. } (\text{arc } AB)^2 : (\text{arc } Ab)^2$$

$$= \text{L. R. } (\text{chord } AB)^2 : (\text{chord } Ab)^2$$

$$= \text{L. R. } (\text{tangent } AD)^2 : (\text{tangent } Ad)^2.$$

COR. 3. Hence if a body describe the arcs AB, Ab with any given velocity, the limiting ratio of the sagittæ will be that of the squares of the times, in which they are described.

Cor. 4. If the subtenses DB, db be perpendicular to the tangent, as in the first case of the Lemma,

$$\triangle ADB : \triangle Adb = AD \cdot DB : Ad \cdot db;$$

$$\begin{aligned}\therefore \text{ L. R. } \triangle ADB : \triangle Adb &= \text{L. R. } AD \cdot DB : Ad \cdot db \\ &= \text{L. R. } AD^3 : Ad^3 \\ \text{or } &= \text{L. R. } DB^{\frac{3}{2}} : db^{\frac{3}{2}}.\end{aligned}$$

Cor. 5. Since L. R. $DB : db =$ L. R. $AD^2 : Ad^2$, the limiting form to which every curve of finite curvature approximates is the common parabola.

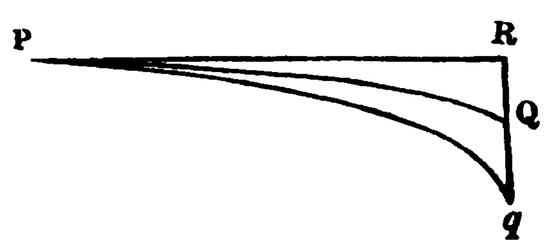
$$\begin{aligned}\text{Hence L. R. area } ADB : \text{area } Adb &= \text{L. R. } \tfrac{2}{3} AD \cdot DB : \tfrac{2}{3} Ad \cdot db \\ &= \text{L. R. } AD^3 : Ad^3 \\ \text{or } &= \text{L. R. } DB^{\frac{3}{2}} : db^{\frac{3}{2}}.\end{aligned}$$

Scholium to Lemma XI.

It was proved in the Lemma that if the curvature be finite, the subtense varies ultimately as the square of the conterminous arc; conversely,

If the subtense vary ultimately as the square of the arc, the curvature is finite, and if it vary according to any other power of the arc, the curvature is infinitely great or infinitely small.

Let PQ and Pq be arcs of a curve and circle, having a common tangent PR, and let RQq be a common subtense.



Since in the circle $qR \propto$ ult. PR^2, let $qR = a \cdot PR^2$ ultimately,

and suppose that $QR \propto$ ult. PR^n and $= b \cdot PR^n$ ultimately

$$\therefore \frac{\text{curvature of } PQ}{\text{curvature of } Pq} = \text{limit } \frac{QR}{qR} = \frac{b}{a} \cdot \text{limit } PR^{n-2}.$$

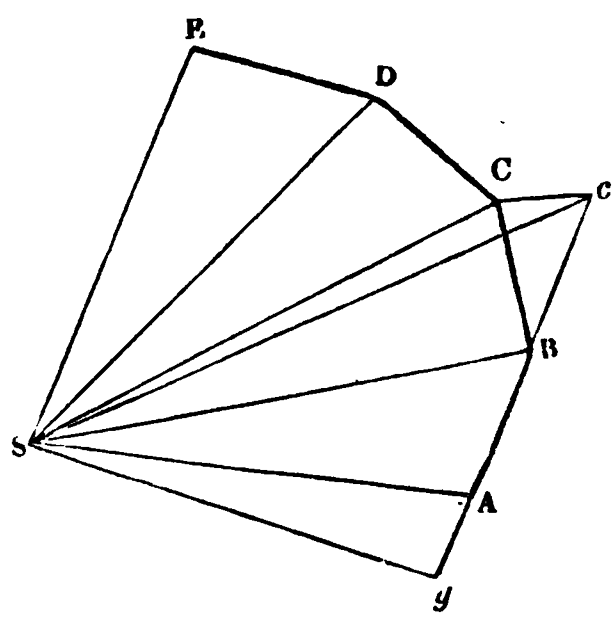
If $n = 2$, the curvature of the curve PQ bears a finite ratio to that of the circle, and is therefore finite. If n be greater than 2, limit $PR^{n-2} = 0$, and therefore the curvature of PQ is infinitely small compared with that of Pq, and the curve will lie between Pq and the tangent. If n be less than 2, limit $PR^{n-2} = \infty$, and therefore the curvature of PQ is infinitely great, and the curve will lie below Pq.

Cor. Since an infinite number of values may be given to n, to each of which there will be a corresponding curve, an infinite number of curves may be described between Pq and the tangent, corresponding to values of n greater than 2, and an infinite number below Pq, corresponding to values of n less than 2.

SECTION II.

ON THE MOTION OF A BODY, CONSIDERED AS A POINT, MOVING IN A NON-RESISTING MEDIUM, AND ATTRACTED TO A SINGLE FIXED CENTER OF FORCE.

PROP. I. *If a body move in any orbit about a fixed center of force, the areas, described by lines drawn from the center to the body, lie in one plane, and are proportional to the times of describing them.*



Let S be the center of force; and suppose a body unattracted by the force in S to describe the straight line AB with a uniform velocity in a given time (T). Then if suffered to proceed, it would move on uniformly in the direction of AB produced, and describe $Bc = AB$ in the next interval (T); but at B suppose an instantaneous impulse communicated to it in direction BS, which causes it to move in direction BC; draw cC parallel to BS, then by the principles of Mechanics, the body at the end of the second interval will be found at C. Join SA, SB, Sc, SC. Since cC is parallel to BS, the triangle $SBC = SBc = SAB$, since $Bc = AB$; and these triangles are in the same plane, as no force has acted to draw the body out of the plane SAB. Similarly, if impulses be communicated at the end of every interval of T'', in directions tending always

to S, causing the body to describe CD, DE, &c. in the third, fourth, &c. intervals, the triangles SAB, SBC, SCD, &c. will be all equal, and will lie in the same plane; and their bases AB, BC, CD, &c. are described in equal times, therefore the area of any number of these triangles, or the polygon $SABCDE$ varies as the time of describing it. Now let the number of intervals be increased, and the magnitude of each diminished indefinitely, then the polygon approximates to a curvilinear area, and the sum of the impulses to a continued force always tending to S, as their limits; and what was proved of those quantities is true of their limits, and therefore the curvilinear area described in any time is proportional to the time.

OBS. The area, described by the line joining S and the body, is frequently called the area described by the body round S.

COR. 1. If V be the velocity of the body at A, and p the perpendicular from S upon the tangent at that point, the area described in $t'' = \frac{1}{2}\, p \,.\, t \,.\, v$.

Draw Sy perpendicular to AB; then since AB is ultimately the tangent at A, limit of $Sy = p$. Also if t be divided into n equal intervals, and AB be the space described in the first interval, the force in S being supposed, as in the Prop., not to act, $AB = \frac{t}{n} \,.\, v$.

Hence, polygonal area described in $t'' = n \,.$ triangle SAB

$$= n \,.\, \tfrac{1}{2} \,.\, Sy \,.\, \frac{t}{n} \,.\, v$$

$$= \tfrac{1}{2}\, Sy \,.\, t \,.\, v\,;$$

and the same is true in the limit,

$$\therefore \text{ curvilinear area described in } t'' = \tfrac{1}{2}\, p \,.\, t \,.\, v.$$

COR. 2. Hence the time of describing any part of the orbit

$$= \frac{2}{p \,.\, v} \,.\, \text{area described.}$$

Cor. 3. If $t = 1$, area described in $1'' = \frac{1}{2} p . v$.

Hence in different orbits, the velocity at any point

$$\propto \frac{\text{area described in } 1''}{\text{perpendicular from } S \text{ upon the tangent}},$$

and in the same orbit, the velocity

$$\propto \frac{1}{\text{perpendicular upon the tangent}}.$$

Prop. II. *If a body, moving in a curve, describe in one plane areas proportional to the times by lines drawn from the body to any point, the body is acted on by centripetal forces all tending to that point.* (Vide Fig. Prop. I.)

Let S be the point, about which areas proportional to the times are described; and suppose as in Prop. 1. that a body, unattracted by the force in S, describes the straight line AB in a given time T.

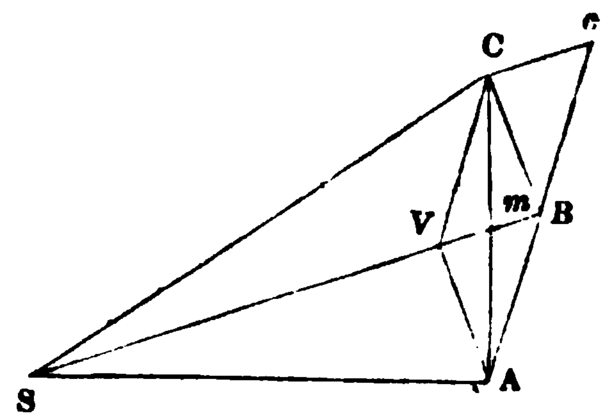
In AB produced take $Bc = AB$; then if suffered to proceed, the body would be at c at end of the second interval of T''. But at B suppose an impulse communicated, which causes it to describe BC in the second interval, such that the triangle $SBC = SAB$. Join cC, Sc.

Then the triangle $SBC = SAB = SBc$, therefore cC is parallel to BS, and therefore by the principles of Mechanics the impulse communicated at B tends to S. Similarly if D, E, &c. be the places of the body at the ends of the third, fourth, &c. intervals of T'', so that the triangles SAB, SBC, SCD, &c. are all equal, all the impulses communicated may be shewn to tend to S.

Now suppose the number of intervals increased, and the magnitude of each diminished indefinitely, then the limit of the

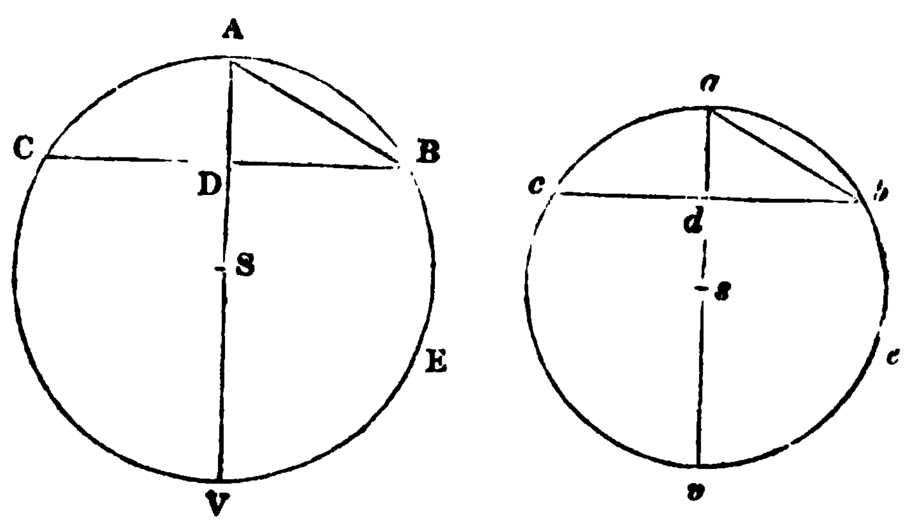
polygon is the curvilinear area and that of the sum of the impulses a continued force tending to S; and the above reasoning still holds in the limit, therefore the body is acted on by a continued force tending to S.

Prop. III. Cor. Draw CV parallel to AB meeting SB in V and join AV. Then $CV = Bc = AB$, $\therefore$ AV is equal and parallel to CB, or $ABCV$ is a parallelogram. Draw the diagonal CA bisecting BV in m.



Now suppose $S'A'B'C'D'$ to be another orbit, in which the chords $A'B'$, $B'C'$ are described in the same time as either of the chords AB or BC; and let the same construction be made as in the former orbit, then impulse at B : impulse at $B' = cC : c'C' = Bm : B'm'$ and therefore force at B : force at B' = L. R. $Bm : B'm'$; or the centripetal forces in different orbits are in the limiting ratio of the sagittæ of arcs described in equal times, which ultimately pass through the centers of force.

Prop. IV. *The centripetal forces, by which bodies describe different circles with uniform velocities, tend to the centers of the circles, and are as the squares of the arcs, described in the same time, divided by the radii.*



Since in each circle the motion is uniform, the arcs described are proportional to the times. But the sectors, i. e. the areas described, are as the arcs on which they stand; and are therefore proportional to the times. Hence (Prop. II.) the forces tend to the centers of the circles.

Again let CAB, cab be arcs described in the same time in the circles, whose centers are S, s, and let A, a be their middle points; join AB, ab, and draw the diameters ASV, asv cutting the chords CB, cb in D, d; then (Prop. II. Cor.)

$$\text{Force at } A : \text{force at } a = \text{L. R. } AD : ad$$

$$= \text{L. R. } \frac{(\text{chord } AB)^2}{AV} : \frac{(\text{chord } ab)^2}{av}$$

$$= \text{L. R. } \frac{(\text{arc } AB)^2}{AS} : \frac{(\text{arc } ab)^2}{as}.$$

Take AE, ae any other arcs described in equal times;

$$\text{then } AE : ae = AB : ab,$$

and this being true whatever be the magnitudes of AB, ab will be true when they are diminished indefinitely,

$$\therefore AE : ae = \text{L. R. } AB : ab,$$

$$\text{and therefore force at } A : \text{force at } a = \frac{AE^2}{AS} : \frac{ae^2}{as}.$$

Cor. 1. Since AE = velocity × time, if V = velocity of the body, R = radius of the circle, and the time be given,

$$F \propto \frac{V^2}{R}.$$

Cor. 2. Let P equal the periodic time, then since $s = tv$,

$$2\pi R = P . V; \;\therefore F \propto \frac{R^2}{P^2 . R} \propto \frac{R}{P^2}.$$

Cor. 3. If P be given, $F \propto R$. If $P \propto R^{\frac{3}{2}}$, $F \propto \frac{1}{R^2}$;

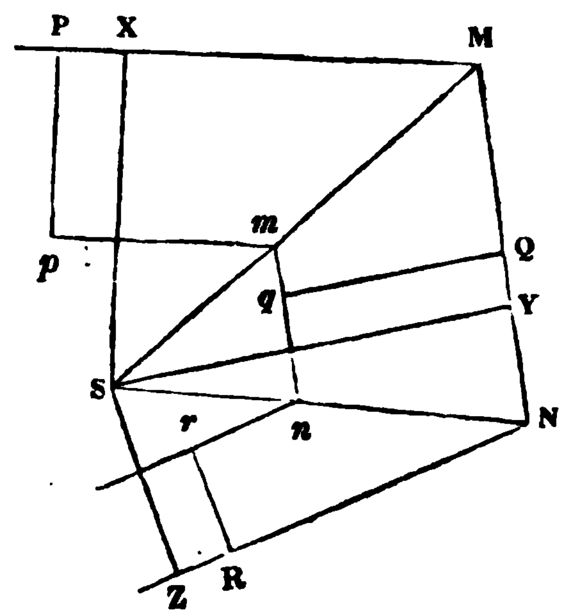
and generally if $P \propto R^n$, $F \propto \frac{1}{R^{2n-1}}$.

PROP. V. *Given the velocities of a body, and the directions of its motion at three points of its orbit, to determine the position of the center of force.*

Let PM, MQN, NR be the directions, in which the body is moving at the three points P, Q, R; draw Pp, Qq, Rr at right angles to these lines respectively, and such that

$$Pp : Qq = \text{velocity at } Q : \text{velocity at } P,$$

$$\text{and } Qq : Rr = \text{velocity at } R : \text{velocity at } Q.$$



Through p, q, r draw pm, mn, nr respectively parallel to PM, MN, NR; join Mm, Nn and produce them to meet in S; S will be the center of force.

Draw SX, SY, SZ perpendicular to PM, MN, NR, respectively,

$$\text{then } \frac{SX}{SM} = \frac{Pp}{Mm},$$

$$\text{and } \frac{SM}{SY} = \frac{Mm}{Qq};$$

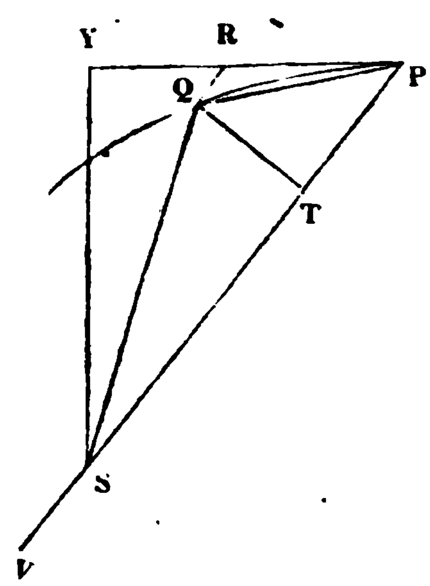
$$\therefore \frac{SX}{SY} = \frac{Pp}{Qq} = \frac{\text{velocity at } Q}{\text{velocity at } P}.$$

$$\text{Similarly } \frac{SY}{SZ} = \frac{\text{velocity at } R}{\text{velocity at } Q}.$$

Hence the perpendiculars, drawn from S upon the tangents at P, Q, R, are inversely as the velocities at those points; therefore S must be the center of force. (Prop. I. Cor. 3.)

Prop. VI. *A body moving round a fixed center of force S, describes the arc PQ in T″; if F be the central force at P, and QR a subtense parallel to SP, when PQ and T are diminished indefinitely,*

$$F = 2\,\text{limit}\,\frac{QR}{T^2}.$$



The motion of the body on leaving P is compounded of two motions, one uniform in direction of the tangent PR, the other variable, arising from the action of S and taking place in direction of the line joining the body with S; therefore since RQ is parallel to PS, its ultimate value, when T is diminished indefinitely, will be the space described by the action of S in that time.

Hence (Lemma x. Cor. 2.) $F = 2\,\text{limit}\,\dfrac{QR}{T^2}$. *

Cor. 1. Draw QT perpendicular to SP, and join SQ, QP; let h = 2 area described in 1″,

$$\text{then } \frac{T''}{1''} = \frac{2\,\text{area}\,PSQ}{h},$$

$$\text{also limit area } PSQ = \text{limit triangle } PSQ$$

$$= \tfrac{1}{2}\,\text{limit}\,QT \cdot SP;$$

* The above expression for the force being obtained independently of the preceding propositions, it is not necessary that the areas described should be proportional to the times. It is therefore true in orbits described round several centers of force, in which case the expression represents the magnitude of the resultant of all the forces acting on the body at the point P. It is clear, however, that the equable description of areas is supposed to be preserved in the three succeeding corollaries. The result in Cor. 4. is general, and might easily be obtained from Cor. 3, in the particular case of the areas being described equably, by substituting for h its value $Sy \cdot V$, obtained in Cor. 2. Prop. I.

$$\therefore \; F = 2 \,\text{limit} \frac{QR}{T^2}$$

$$= 2 \frac{h^2}{4} \,.\, \text{limit} \frac{QR}{\frac{1}{4} QT^2 . SP^2}$$

$$= \frac{2h^2}{SP^2} \,.\, \text{limit} \frac{QR}{QT^2}.$$

Cor. 2. Draw Sy perpendicular to the tangent PR, then since the angle QPR ultimately vanishes, the triangles QPT, SPy are ultimately similar;

$$\therefore \; \text{limit} \frac{QT}{PQ} = \frac{Sy}{SP},$$

$$\therefore \; \text{limit} \frac{QR}{QT^2} = \frac{SP^2}{Sy^2} \text{limit} \frac{QR}{PQ^2},$$

$$\therefore \; F = \frac{2h^2}{Sy^2} \text{limit} \frac{QR}{PQ^2}.$$

Cor. 3. If PV be the chord of curvature at P through S,

$$PV = \text{limit} \frac{PQ^2}{QR}, \quad \therefore \; F = \frac{2h^2}{Sy^2 . PV}.$$

Obs. If A = the area described in P'', $h = \frac{2A}{P}$, which value may be substituted for h in the above expressions for the force.

Cor. 4. *The space, through which a body must descend from rest by the action of the force at P continued constant, in order to acquire the velocity at P, is $\frac{1}{4}$th of the chord of curvature PV.*

Since limit $\frac{PR}{PQ} = 1$, $\quad F = 2 \,\text{limit} \frac{QR}{T^2} = 2 \,\text{limit} \frac{QR}{PQ^2} . \left(\frac{PR}{T}\right)^2$

Now limit $\frac{QR}{PQ^2} = \frac{1}{PV}$, and limit $\frac{PR}{T}$ = velocity at $P = V$;

$$\therefore F = \frac{2V^2}{PV} \text{ and therefore, } V^2 = F \cdot \frac{PV}{2}.$$

Let S = space due to V by the action of F continued constant,

$$\text{then, } V^2 = 2FS,$$

hence equating this to the above expression for V^2, we have

$$S = \tfrac{1}{4} PV.$$

Cor. 5. To find the velocity and periodic time of a body, revolving in a circle and acted on by a centripetal force tending to the center of the circle.

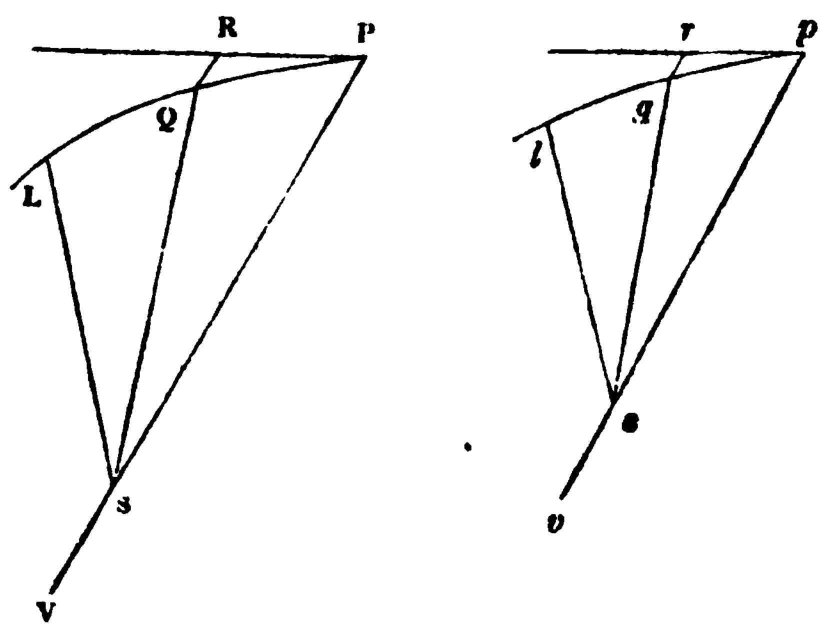
Here PV = the diameter $= 2R$, $\therefore v = \sqrt{F \cdot R}$;

$$\text{Also } P = \frac{\text{circumference}}{\text{velocity}} = \frac{2\pi R}{\sqrt{F \cdot R}} = 2\pi \sqrt{\frac{R}{F}}.$$

Cor. 6. If V, v be the velocities at P, p, points similarly situated in similar orbits, described round S, s centers of force, also similarly situated,

$$\text{Force at } P\,(F) : \text{force at } p\,(f) = \frac{V^2}{SP} : \frac{v^2}{sp}.$$

Let PQ, pq be arcs described in equal times, QR, qr subtenses parallel to SP, sp, and PV, pv chords of curvature at P, p through S, s.



Then since the times are equal,

$$F : f = \text{L. R. } QR : qr$$

$$= \text{L. R. } \frac{PQ^2}{PV} : \frac{pq^2}{pv},$$

$$\text{also } V : v = \text{L. R. } \frac{PQ}{T} : \frac{pq}{T}$$

$$= \text{L. R } PQ : pq,$$

and since P and p are points similarly situated in similar orbits,

$$SP : sp = PV : pv,$$

$$\therefore F : f = \frac{V^2}{SP} : \frac{v^2}{sp}.$$

Cor. 7. If similar arcs of similar orbits be described in times T, t round S, s, centers of force similarly situated, (Fig. Cor. 6.)

$$F : f = \frac{SP}{T^2} : \frac{sp}{t^2}.$$

Let PL, pl be similar arcs described in times T, t, and take PQ, pq other similar arcs described in times P, p; QR, qr subtenses parallel to SP, sp; then

$$F : f = \text{L. R. } \frac{QR}{P^2} : \frac{qr}{p^2},$$

join SQ, SL, sq, sl.

Then $T : P = \text{area } PSL : \text{area } PSQ$

$= \text{area } psl : \text{area } psq$, by similar figures

$= t : p$,

$$\therefore T : t = P : p;$$

and this, being always true, will be true when P and p are diminished indefinitely,

$$\therefore\ T : t = \text{L.R.}\ P : p,$$

and by similar figures,

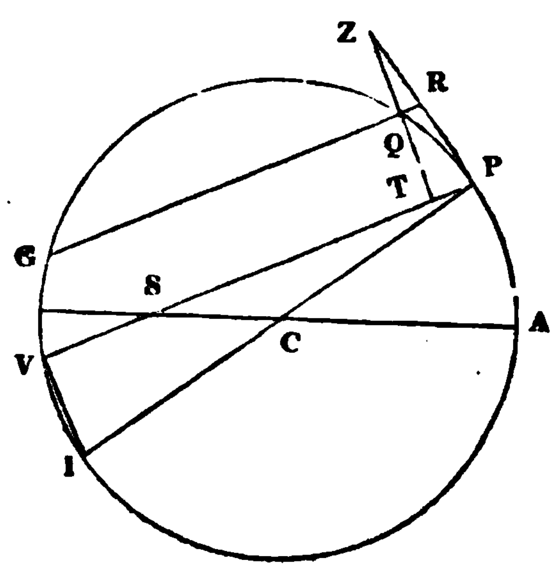
$SP : sp = QR : qr$ always and therefore in

the limit;

$$\therefore\ F : f = \frac{SP}{T^2} : \frac{sp}{t^2}.$$

PROP. VII. *A body revolves in the circumference of a circle, to find the law of force by which it is attracted to a given point.*

Let PAV be the circumference of the circle, and S the center of force; PQ a small arc, QR a subtense parallel to SP, QT perpendicular to SP. Let RQ, and PS produced if necessary, meet the circumference in G, V; draw the diameter PI, join IV, and produce TQ, PR to meet in Z. The triangles PTZ, PVI are evidently similar.



$$\text{Hence } \frac{QR \cdot RG}{QT^2} = \frac{RP^2}{QT^2} \text{ (Euc. III. 36.)} = \frac{ZP^2}{ZT^2} = \frac{PI^2}{PV^2}.$$

Now let Q move up to P,

$$\text{then limit } \frac{QR}{QT^2} = \text{limit } \frac{PI^2}{PV^2 \cdot RG}$$

$$= \text{limit } \frac{PI^2}{PV^3}, \text{ since limit } RG = PV.$$

$$\therefore \; F = \frac{2h^2}{SP^2} \,.\, \text{limit} \; \frac{QR}{QT^2}$$

$$= \frac{2h^2}{SP^2} \cdot \frac{PI^2}{PV^3} = \frac{8h^2R^2}{SP^2 \,.\, PV^3},$$

if R = radius of the circle.

Let μ represent that part of the expression for F, which in the same orbit is invariable; then in this case,

$$\mu = 8h^2 R^2,$$

$$\text{Hence} \quad F = \frac{\mu}{SP^2 \,.\, PV^3},$$

and therefore in the same circle $\propto \dfrac{1}{SP^2 \,.\, PV^3}$.

Cor. 1. To find the velocity at any point.

$$V^2 = F \,.\, \frac{PV}{2} = \frac{4h^2 \,.\, R^2}{SP^2 \,.\, PV^2}, \quad \text{or} = \frac{\mu}{2 \,.\, SP^2 \,.\, PV^2};$$

$$\therefore \; v = \frac{2hR}{SP \,.\, PV}, \quad \text{or} = \sqrt{\frac{\mu}{2}} \cdot \frac{1}{SP \,.\, PV}.$$

Obs. The quantity (μ) here introduced is that part of the general expression for the centripetal force in any orbit, which is invariable for all points in that orbit, and may always be determined, if the actual force at any given point be known. The force, by which a body is retained in a given curve, is in most cases undergoing a continual change in magnitude; but its magnitude at any given point is to be estimated by the effect it would produce, that is, by the velocity it would generate in a unit of time from rest, supposing it to remain constant for that time. Hence if a second and a foot be the units of time and space, the magnitude of the centripetal force at any

point is represented by twice the number of feet, which it would cause a body to describe from rest in 1″; if for instance, it draws a body from rest through 10 feet in 1″, its magnitude will be 20, and it will be to the force of gravity in the ratio of 20 : $32\frac{1}{5}$ or of 100 : 161. Suppose then in the preceding proposition, that the force at A, the extremity of the diameter through S, would if continued constant draw a body through (f) feet in 1″;

$$\text{then } \frac{\mu}{SA^2 \,.\, (2R)^3} = 2f;$$

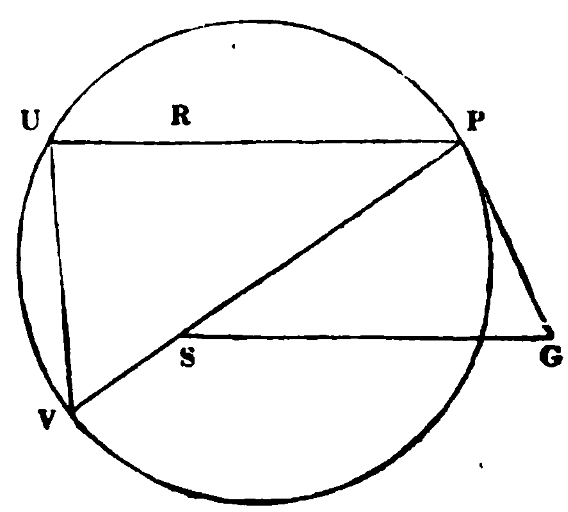
$$\therefore \; \mu = 2f \,.\, SA^2, (2R)^3.$$

Cor. 2. Let S be in the circumference, then $PV = SP$.

$$\text{Hence } F = \frac{8h^2R^2}{SP^5}, \quad \text{or} = \frac{\mu}{SP^5}; \text{ and therefore, } \propto \frac{1}{SP^5},$$

$$V = \frac{2hR}{SP^2}, \quad \text{or} = \sqrt{\frac{\mu}{2}} \,.\, \frac{1}{SP^2}.$$

Cor. 3. To compare the forces, by which a body, attracted separately to two centers of force, may describe the same circle in the same periodic time.



Let R and S be the two centers of force; produce PR, PS if necessary to meet the circumference in U, V; draw SG parallel to RP to meet the tangent at P in G, and join UV; then the triangles SPG, PVU are evidently similar,

$$\therefore \frac{SG}{SP} = \frac{PV}{PU}, \quad \text{or } SG = \frac{PV \,.\, SP}{PU}.$$

Also since the periodic time is the same, h, which $= \frac{2 \text{ area of circle}}{\text{periodic time}}$, is the same for both centers, hence

$$F \text{ to } R : F \text{ to } S = \frac{1}{RP^2 . PU^3} : \frac{1}{SP^2 . PV^3}$$

$$= \frac{PV^3 . SP^3}{PU^3} : RP^2 . SP$$

$$= SG^3 : RP^2 . SP.$$

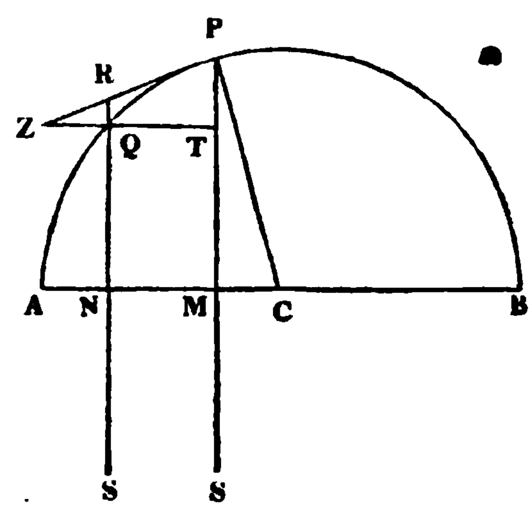
Cor. 4. What has been proved in the last corollary in the case of the circle is true of any orbit described round two centers of force separately in the same periodic time. For if PUV be the circle of curvature at P, the expression for F, viz. 2 limit $\frac{QR}{QT^2}$, is the same in the curve and circle, and therefore what has been proved in the one case is true in the other. Hence generally in any orbit described in the same time round two centers of force,

$$F \text{ to } R : F \text{ to } S = SG^3 : RP^2 . SP.$$

If the periodic times are not the same,

$$F \text{ to } R : F \text{ to } S = \frac{SG^3}{P^2 \text{ round } R} : \frac{RP^2 . SP}{P^2 \text{ round } S}.$$

Prop. VIII. *To find the law of force by which a body may describe a semicircle, the center of force being so distant, that all lines drawn from it to the body may be considered parallel.*



Let PQ be a small arc of the semicircle, C the center; draw PS, QS parallel to each other toward the center of force; CM perpendicular to PS, then CM produced both ways will determine the semicircle described. Draw QT perpendicular, and QR parallel to SP, and produce PR, TQ to meet in Z; join CP. The triangles PZT, CPM are evidently similar;

$$\therefore \frac{QR \cdot (RN + QN)}{QT^2} = \frac{RP^2}{QT^2} = \frac{ZP^2}{ZT^2} = \frac{CP^2}{PM^2};$$

$$\therefore \text{ limit } \frac{QR}{QT^2} = \frac{CP^2}{2PM^3}, \text{ since limit } (RN + QN) = 2PM;$$

$$\therefore F = \frac{2h^2}{SP^2} \cdot \text{limit } \frac{QR}{QT^2}$$

$$= \frac{h^2 \cdot CP^2}{SP^2 \cdot PM^3}, \text{ or } = \frac{\mu}{PM^3}, \text{ and } \therefore \propto \frac{1}{PM^3}.$$

Cor. To find the velocity at any point.

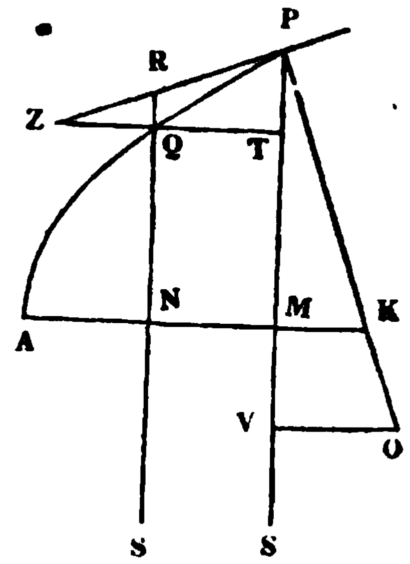
$$V^2 = F \cdot \frac{PV}{2} = \frac{h^2 \cdot CP^2}{SP^2 \cdot PM^3} \cdot PM$$

$$= \frac{h^2 \cdot CP^2}{SP^2 \cdot PM^2},$$

$$\therefore V = \frac{h \cdot CP}{SP \cdot PM}, \text{ or } = \frac{\sqrt{\mu}}{PM}.$$

Scholium to Prop. VIII.

If AQP be any conic section, it may be described by the action of a force tending to a point at an infinite distance, and varying inversely as the cube of the ordinate.



Let PO, the diameter of curvature at P, cut the axis of the conic section in K; draw OV perpendicular to PS, then PV is the chord of curvature at P in direction of the force; and complete the construction as in the proposition.

By similar triangles ZPT, PMK,

$$\frac{QT^2}{QR} : \frac{RP^2}{QR} = ZT^2 : ZP^2 = PM^2 : PK^2,$$

and this being true always will be true, when Q moves up to P,

$$\therefore \text{ L. R. } \frac{QT^2}{QR} : \frac{RP^2}{QR} = PM^2 : PK^2,$$

$$\text{and } PV : PO = PM : PK,$$

$$\therefore \text{ since limit } \frac{RP^2}{QR} = \text{limit } \frac{PQ^2}{QR} = PV,$$

$$\text{limit } \frac{QT^2}{QR} : PO = PM^3 : PK^3.$$

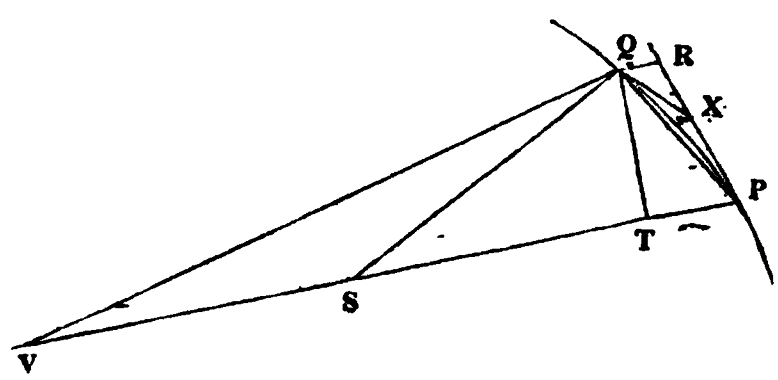
Now (Appendix Arts. 4, 5.) in all conic sections, the diameter of curvature $= \dfrac{8}{L^2} . PK^3$,

$$\therefore \text{ limit } \frac{QT^2}{QR} = \frac{8\,PM^3}{L^2},$$

$$\text{and } \therefore F = \frac{h^2 . L^2}{4SP^2 . PM^3} \propto \frac{1}{PM^3}.$$

Prop. IX. *To find the law of force tending to the pole, by which a body may describe an equiangular spiral.*

Def. An equiangular spiral, is a spiral cutting all the radii at the same given angle.



Let PQ be a small arc of the spiral, S the center of force in the pole. QR a subtense parallel to SP, QT perpendicular to SP, and let the constant angle SPR, which the curve makes with the radius, $= a$. Join PQ, and let PV be the chord of curvature through S,

$$\text{then } \frac{QT^2}{QR} = \frac{PR^2}{QR}\sin^2 a\,;$$

$$\therefore \text{ limit } \frac{QT^2}{QR} = \text{limit } \frac{PR^2}{QR}\sin^2 a = \text{limit } \frac{PQ^2}{QR}\sin^2 a = PV\sin^2 a.$$

Let the tangent at Q intersect PR in X. Then since SP, SQ make equal angles with the tangents at P, Q, the angles SPX, SQX are equal to two right angles, therefore the angle $PSQ =$ angle QXR. Also since V is a point in the circumference of the circle of curvature, the angles XPQ, XQP are each ultimately equal to QVB. Hence the angle QXR, and therefore the angle QSP is ultimately double of the angle QVS, therefore $\angle\, SQV$ is ultimately equal to $\angle\, SVQ$, or $SV = SQ$ ultimately $= SP$. Hence $PV = 2SP$,

$$\text{and } \therefore F = \frac{2h^2}{SP^2}\text{ limit } \frac{QR}{QT^2}$$

For $PXQ = \pi - XSQ = \pi - SQR$

$QXR = \pi - (XQR + QRX) = \pi - SQR$

$$= \frac{2h^2}{SP^2} \cdot \frac{1}{2SP \sin^2 \alpha}$$

$$= \frac{h^2}{\sin^2 \alpha} \cdot \frac{1}{SP^3}, \text{ or } = \frac{\mu}{SP^3}, \text{ and } \therefore \propto \frac{1}{SP^3}.$$

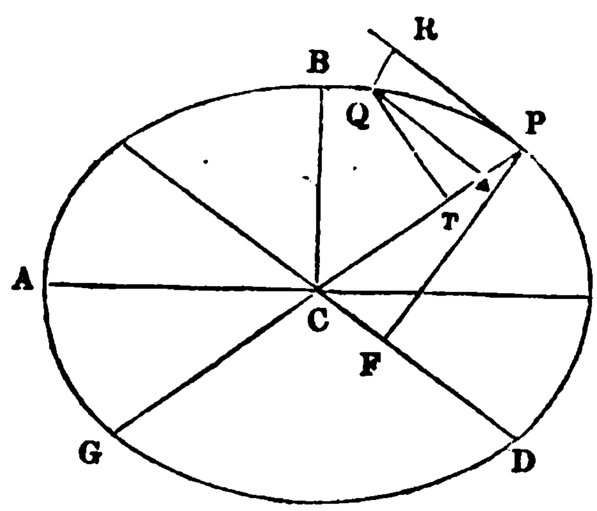
Cor. To find the velocity at any point.

$$V^2 = F \cdot \frac{PV}{2} = \frac{h^2}{\sin^2 \alpha} \cdot \frac{1}{SP^2}, \text{ or } \frac{\mu}{SP^2};$$

$$\therefore V = \frac{h}{\sin \alpha} \cdot \frac{1}{SP}, \text{ or } \frac{\sqrt{\mu}}{SP}.$$

Prop. X. *A body describes an ellipse round a center of force in the center of the ellipse, to find the law of force.*

Let PQ be a small arc of the ellipse, C the center, QR a subtense parallel to CP; AC, BC the semi-axes major and minor; QV parallel to PR; QT, PF perpendicular to CP and the semi-conjugate CD respectively, produce PC to meet the ellipse again in G; then the triangles QVT, PCF are evidently similar.



$$\text{Now } \frac{PV \cdot VG}{QV^2} = \frac{CP^2}{CD^2},$$

$$\text{and } \frac{QV^2}{QT^2} = \frac{CP^2}{PF^2};$$

$$\therefore \frac{PV \cdot VG}{QT^2} = \frac{CP^4}{PF^2 \cdot CD^2} = \frac{CP^4}{AC^2 \cdot BC^2};$$

$$\therefore \text{ limit } \frac{QR}{QT^2} = \text{limit } \frac{PV}{QT^2} = \frac{CP^4}{AC^2 . BC^2 . 2CP},$$

(since limit $VG = 2CP$)

$$= \frac{CP^3}{2AC^2 . BC^2};$$

$$\therefore \; F = \frac{2h^2}{CP^2} \text{ limit } \frac{QR}{QT^2} = \frac{h^2}{AC^2 . BC^2} . CP,$$

or $= \mu . CP$, and therefore $\propto CP$.

COR. 1. To find the velocity at any point.

$$V^2 = \tfrac{1}{2} F . PV = \tfrac{1}{2} \frac{h^2}{AC^2 . BC^2} CP . \frac{2CD^2}{CP}, \text{ since } PV = \frac{2CD^2}{CP}$$

$$= \frac{h^2}{AC^2 . BC^2} CD^2;$$

$$\therefore \; V = \frac{h}{AC . BC} . CD, \text{ or } \sqrt{\mu} . CD.$$

COR. 2. To find the periodic time.

Since $\mu = \dfrac{h^2}{AC^2 . BC^2}$, $h = AC . BC . \sqrt{\mu}$;

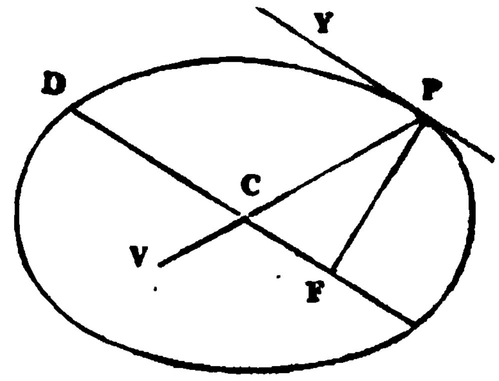
also the area of the ellipse $= \pi AC . BC$;

$$\therefore \; P = \frac{2 \text{ area of ellipse}}{h}$$

$$= \frac{2\pi}{\sqrt{\mu}}.$$

Hence the periodic times in all ellipses round the same center of force in the center are equal.

Cor. 3. *If a body be projected in a direction making any angle with its distance from a fixed point, and be attracted to that point by a force varying as the distance, it will describe an ellipse, whose center is the center of force.*



Let C be the center of force, P the point from which the body is projected in direction PY, V the velocity, and F the force at P.

Then space (s) due to the velocity at $P = \dfrac{V^2}{2F}$. In PC, produced if necessary, take $PV = 4s$, and draw CD parallel to PY and $= \sqrt{\frac{1}{2} CP . PV}$. With CP, CD as semi-conjugate diameters describe an ellipse, and suppose a body revolving in it to come to P; then it is moving in the direction of the tangent at P, that is, in a line parallel to CD or in direction PY. Also space due to velocity at $P = \frac{1}{4}$ chord of curvature at P

$$= \frac{1}{4} \cdot \frac{2\,CD^2}{CP} = \frac{1}{4}\,PV = s.$$

The force, distance, and law of force are the same also in both cases; hence the two bodies are under the same circumstances at P, and will therefore describe the same orbit; that is, the projected body will describe an ellipse, whose center is C.

If CPY be a right angle, and $s = \frac{1}{2} PC$, the orbit described will be a circle.

Cor. 4. To compare the velocity at P with the velocity in a circle, radius $= CP$, described round the same center of force.

$$V. \text{ in ellipse} = \sqrt{\mu} \, . \, CD.$$

$$V. \text{ in circle (radius} = CP) = \sqrt{F . CP}, \text{ (Prop. vi. Cor. 5.)}$$
$$= \sqrt{\mu} \, . \, CP;$$

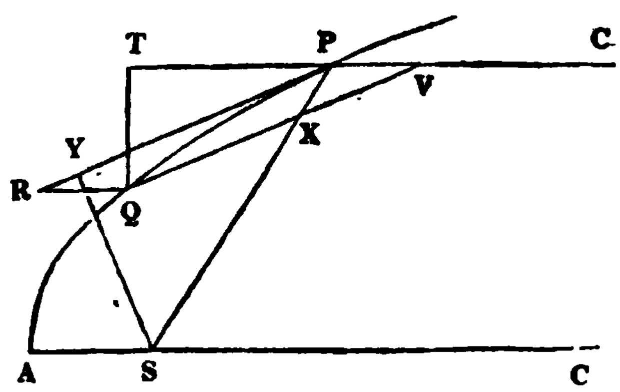
$$\therefore \; V. \text{ in ellipse} : V. \text{ in circle (rad.} = CP) = CD : CP.$$

Scholium to Prop. X.

1. It was proved in the proposition, that, when a body moves in an ellipse round a center of force in the center, the force varies as dist. The same is also true, when a body moves in an hyperbola, the construction and proof being exactly the same as for the ellipse.

2. If the orbit be a parabola, the center of force is removed to an infinite distance, and the force acts in lines parallel to the axis; in this case, since the difference of any two distances vanishes compared with the distances themselves, the force is invariable.

Or the following proof may be applied in the case of the parabola.



Let PQ be a small arc of the parabola, A the vertex, S the focus; PC parallel to the axis, and therefore in the direction of the force; QR a subtense parallel to PC, and QV parallel to the tangent PR; QT, SY perpendicular to CP, PR.

Since $4SP \cdot PV = QV^2$, $\dfrac{QR}{QV^2} = \dfrac{PV}{QV^2} = \dfrac{1}{4SP}$,

and by similar triangles, QTV, SPY,

$$\frac{QV^2}{QT^2} = \frac{SP^2}{SY^2} = \frac{SP}{SA};$$

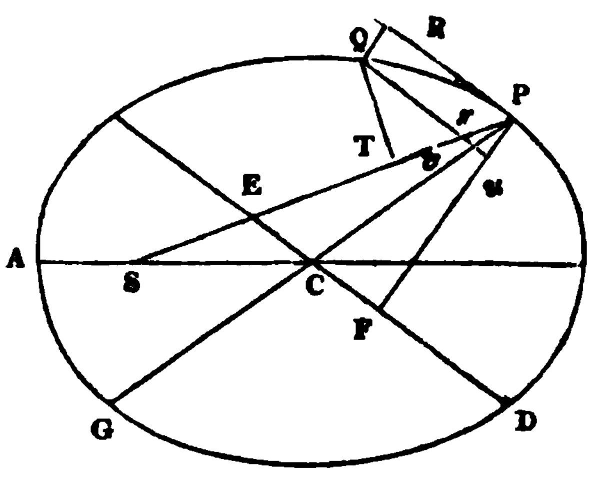
$$\therefore \text{ limit } \frac{QR}{QT^2} = \frac{1}{4SA};$$

$$\therefore F = \frac{2h^2}{CP^2} \cdot \frac{1}{4SA}.$$

SECTION III.

ON THE MOTION OF BODIES IN CONIC SECTIONS, ABOUT A CENTER OF FORCE IN ONE OF THE FOCI.

Prop. XI. *A body revolves in an ellipse, to find the law of force tending to one of the foci.*



Let the focus S be the center of force, PQ a small arc; QR a subtense parallel to SP; C the center of the ellipse, join PC and produce it to meet the ellipse in G; draw Qxv parallel to the tangent PR, cutting SP, CP in x, v; and QT, PF respectively perpendicular to SP, and the semi-conjugate diameter CD: and let E be the point, in which SP cuts CD, then $PE = AC$, the $\frac{1}{2}$ axis major.

By similar triangles, QxT, PEF,

$$\frac{Qx^2}{QT^2} = \frac{PE^2}{PF^2} = \frac{AC^2}{PF^2},$$

and by a property of the ellipse,

$$\frac{Pv}{Qv^2} = \frac{CP^2}{vG \cdot CD^2};$$

also by similar triangles, PxV, PEC,

$$\frac{Px}{Pv} = \frac{PE}{PC} = \frac{AC}{PC}.$$

Now $Px = QR$, Qx ultimately $= Qv$, and vG ultimately $= 2CP$; hence multiplying the above quantities together, and taking the limits of the products,

$$\text{limit } \frac{QR}{QT^2} = \frac{AC^3 \cdot CP}{2CP \cdot CD^2 \cdot PF^2} = \frac{AC^3}{2AC^2 \cdot BC^2},$$

$$= \frac{AC}{2BC^2} = \frac{1}{L};$$

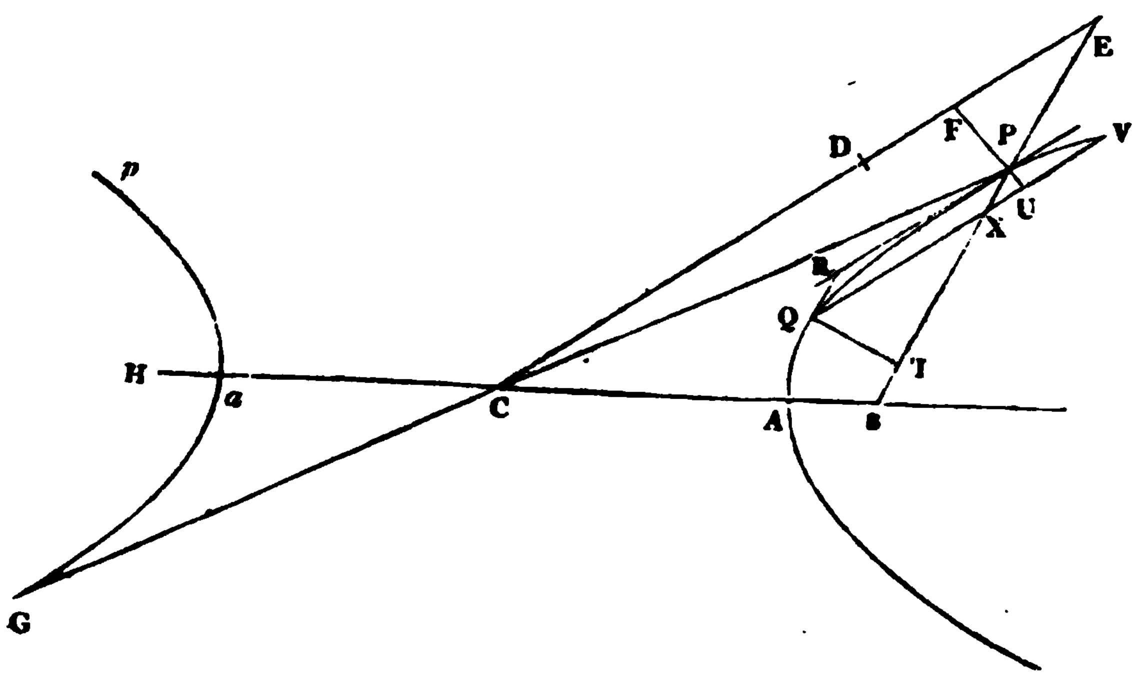
$$\therefore F = \frac{2h^2}{SP^2} \text{ limit } \frac{QR}{QT^2}$$

$$= \frac{2h^2}{L} \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2}$$

$$\propto \frac{1}{SP^2}.$$

Prop. XII. *A body moves in an hyperbola, to find the law of force tending to one of the foci.*

Let the center of force be in the focus S, and let the body move in the branch PA, which is nearer to S than

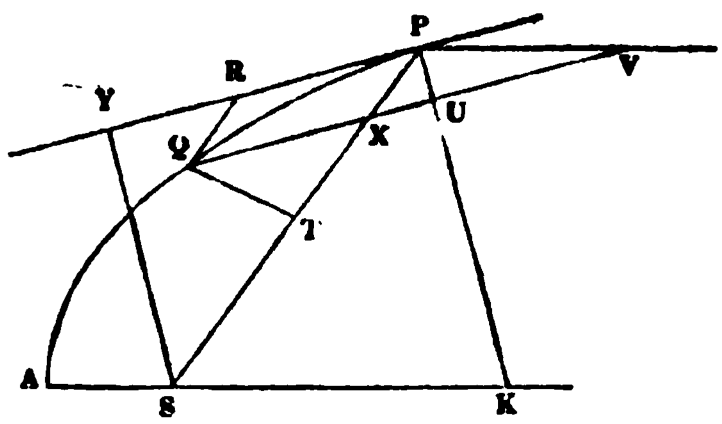


the other branch of the hyperbola. Then the same construction being made as in the ellipse, it may be shewn in precisely the same manner that the force

$$= \frac{2h^2}{L} \cdot \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2} \text{ and } \therefore \propto \frac{1}{SP^2}.$$

COR. In the same manner it may be shewn, that if the body describes the opposite branch pa by a *repulsive* force proceeding from S, the force will vary as $\frac{1}{Sp^2}$.

PROP. XIII. *A body moves in a parabola, to find the law of force tending to the focus.*



Let A and S be the vertex and focus of the parabola, PQ a small arc; QR a subtense parallel to SP, QXV parallel to the tangent PR, cutting SP in X, and the diameter through P in V; QT, SY perpendicular to SP, PR; $L =$ latus rectum. Then by a property of the parabola $PV = PX$ and $\therefore = QR$; also $4SP \cdot PV = QV^2$.

Hence

$$\frac{QR}{QV^2} = \frac{PV}{QV^2} = \frac{1}{4SP},$$

and by similar triangles,

$$\frac{QX^2}{QT^2} = \frac{SP^2}{SY^2} = \frac{SP}{SA}.$$

Now QX ultimately $= QV$; hence multiplying these quantities together, and taking the limits,

$$\text{limit } \frac{QR}{QT^2} = \frac{1}{4SA} = \frac{1}{L},$$

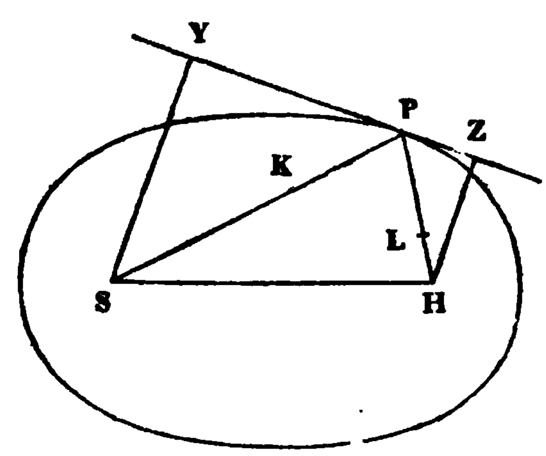
$$\therefore\ F = \frac{2h^2}{SP^2} \text{ limit } \frac{QR}{QT^2},$$

$$= \frac{2h^2}{L} \cdot \frac{1}{SP^2} \text{ or } = \frac{\mu}{SP^2},$$

$$\propto \frac{1}{SP^2}.$$

Cor. *If a body be projected at a given distance from a center of force, which $\propto (dist.)^{-2}$, and in a direction making a finite angle with the distance, it will describe a conic section.*

Let S be the center of force, P the point and PY the direction of projection, $F=$ the force at P, then if s be the space due to the velocity of projection, $s = \dfrac{(\text{velocity})^2}{2F}$ and is therefore known.

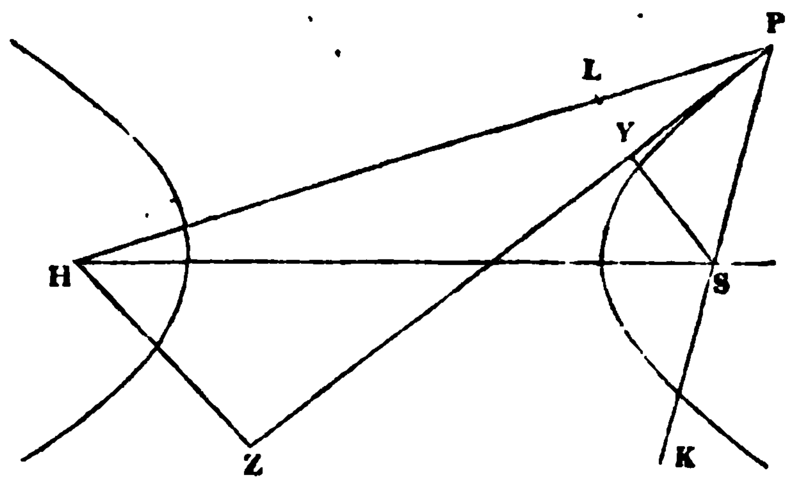


1. Let s be less than SP. In PS take $PK = s$, and draw PH, making with YP produced the $\angle HPZ = \angle SPY$; in PH take $PL = SK$, and through S, K, L describe a circle cutting PL in H; so that $PH.PL = PS.PK$. With foci S and H and axis major $= SP + HP$, describe an ellipse, and suppose a body revolving in this ellipse and acted on by the same force in S, to come to P; then space due to velocity at $P = \frac{1}{4}$ chord of curvature at P through S,

$$= \frac{(\frac{1}{2}\text{ conjugate diameter})^2}{\text{axis major}} = \frac{SP.HP}{SP+HP},$$

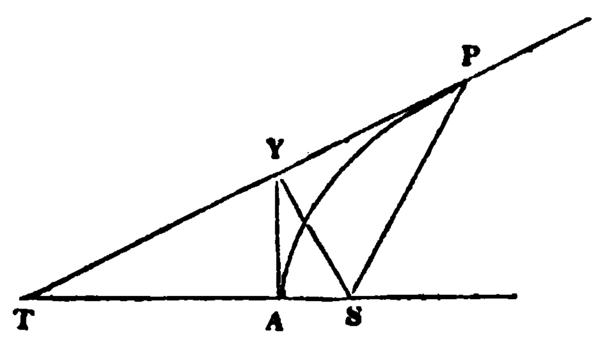
$$= \frac{SP}{1+\dfrac{SP}{HP}} = \frac{SP}{1+\dfrac{PL}{PK}} = PK.$$

Hence the velocity is the same in both cases; also the revolving body is moving in direction PY, since ZPy, making equal angles with SP, HP, is a tangent at P; and the force and the law of force are the same for both bodies; they will therefore describe the same curve, that is, the projected body will describe an ellipse.



2. Let s be greater than SP. In PS produced take $PK = s$; draw PH on the other side of PY, making the $\angle YPH = \angle YPS$, take $PL = SK$, and through S, K, L, describe a circle cutting PL produced in H: then if with foci S and H and axis major $= HP \sim SP$, an hyperbola be described, it may be shewn, as in the preceding case, that the body will move in the hyperbola thus constructed.

3. Let $s = SP$. Here $SK = 0$, and $\therefore PH = \dfrac{PK \cdot PS}{SK} = \infty$.



Let the circle described with center S and radius SP cut PY in T; draw SY, YA perpendicular to PT, TS respectively, and with focus S and vertex A, describe a parabola; then it may be shewn as in the former cases, that the body will move in the parabola thus constructed.

Prop. XIV. *If any number of bodies revolve about one common center of force, which varies as* $(dist.)^{-2}$, *and is the same at equal distances in all the orbits described, the latera recta of the orbits will be as the squares of the areas described in equal times.*

Let $\dfrac{\mu}{SP^2}$ be the force in any orbit at the distance SP, then since the forces at equal distances are equal, μ is the the same for all the orbits:

Also by Props. XI, XII, XIII, $\mu = \dfrac{2h^2}{L}$,

$$\therefore L \propto h^2 \propto \left(\frac{\text{area described in a given time}}{\text{time}}\right)^2$$

$$\propto (\text{area})^2 \text{ described in a given time.}$$

Prop. XV. ***A body revolves in an ellipse round a center of force in the focus, to find the periodic time.***

Let AC, BC be the semi-axes major and minor, P the periodic time.

Then $\dfrac{P''}{1''} = \dfrac{\text{area of the ellipse}}{\text{area described in } 1''}$,

$$\therefore P = \frac{\pi AC \cdot BC}{\frac{1}{2}h},$$

and since $\dfrac{2h^2}{L} = \mu$, $h = \sqrt{\dfrac{\mu L}{2}} = \sqrt{\dfrac{\mu \cdot BC^2}{AC}} = BC\sqrt{\dfrac{\mu}{AC}}$,

$$\therefore P = \frac{2\pi AC^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}.$$

Cor. Hence, the squares of the periodic times in all ellipses, described round the same center of force in the focus, are as the cubes of the major axes.

Prop. XVI. ***To find the velocity at any point of a conic section, described about a center of force in the focus.***

Let V be the velocity at the point P,

$$V^2 = \tfrac{1}{2}F \cdot PV = \frac{\mu}{SP^2} \cdot \frac{PV}{2}.$$

Now in the ellipse and hyperbola,

$$\frac{PV}{2} = \frac{CD^2}{AC} = \frac{SP \cdot HP}{AC} = SP \cdot \left(2 \mp \frac{SP}{AC}\right),$$

and in the parabola,

$$\frac{PV}{2} = 2SP.$$

Hence in the ellipse $V = \sqrt{\frac{\mu}{SP}\left(2 - \frac{SP}{AC}\right)}$,

in hyperbola $V = \sqrt{\frac{\mu}{SP}\left(2 + \frac{SP}{AC}\right)}$,

in parabola $V = \sqrt{\frac{2\mu}{SP}}$.

Cor. To compare the velocity with that of a body moving in a circle, radius $= SP$, and described round the same centre of force.

Let $U =$ velocity in the circle,

then (Prop. vi. Cor. 5.),

$$U = \sqrt{F . R} = \sqrt{\frac{\mu}{SP^2} SP} . = \sqrt{\frac{\mu}{SP}},$$

$\therefore$ in ellipse $\frac{V}{U} = \sqrt{2 - \frac{SP}{AC}}$ which is less than $\sqrt{2}$,

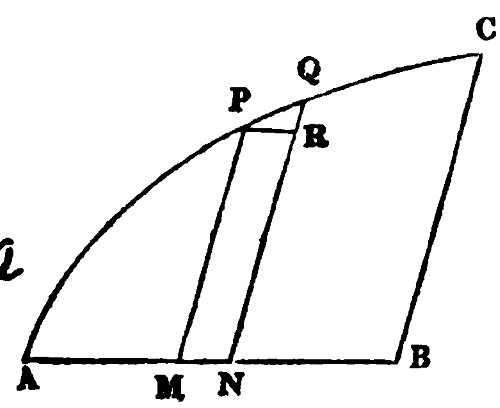
in hyperbola $\frac{V}{U} = \sqrt{2 + \frac{SP}{AC}}$......... greater ....

in parabola $\frac{V}{U} = \sqrt{2}$.

APPENDIX.

NOTE TO LEMMA II.

1. To find the area of a plane curve.



Let the area ABC be bounded by the curve AC, and the straight lines AB, BC. Let AB be divided into n equal parts, and let MN be the r^{th} part from A; draw MP, NRQ parallel to BC, and complete the parallelogram $MNRP$.

Let $AB = h$, then $MN = \frac{h}{n}$,

$MP = y_r$,

$\angle ABC = i$,

$$\text{area of parallelogram } PN = \frac{h}{n} y_r \sin i.$$

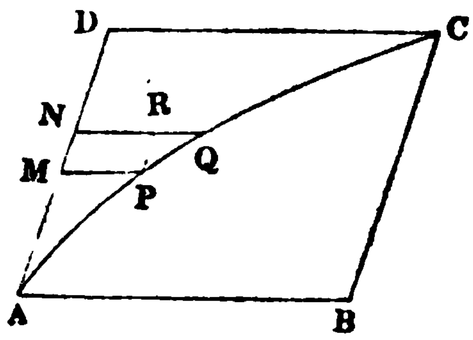
Therefore giving to r the values $1, 2, 3 \ldots n$, the sum of the parallelograms described on all the parts

$$= \frac{h}{n} \sin i \, (y_1 + y_2 + y_3 + \ldots + y_n) = h \sin i \, . \, \Sigma \, . \, \frac{y_r}{n}.$$

Therefore area of curvilinear figure $= h \sin i \, . \, \text{limit} \, \Sigma \, \frac{y_r}{n}$, when n is infinite.

Ex. 1. To find the area of a portion of a parabola cut off by a diameter, and one of its ordinates.

Let ABC be the parabolic area cut off by the diameter AB and a semi-ordinate BC. Complete the parallelogram $ABCD$: then AD is a tangent at A.



Let $AD = h$, $AB = k$, and let AM be the abscissa, and MP, parallel to AB, the ordinate to the point P; then by a property of the parabola,

$$\frac{PM}{AM^2} = \frac{AB}{AD^2}; \quad \therefore PM \text{ or } y_r = \frac{k}{h^2} \cdot \left(\frac{rh}{n}\right)^2 = k\frac{r^2}{n^2},$$

$$\therefore \text{ area } ADC = h \sin i \,.\, \text{limit} \,.\, \Sigma\frac{y_r}{n} = kh \sin i \text{ limit } \Sigma\frac{r^2}{n^3},$$

$$= hk \sin i \,.\, \text{limit} \,.\, \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \dots + n^2),$$

$$= hk \sin i \,.\, \text{limit } \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \tfrac{1}{3} hk \sin i$$

$$= \tfrac{1}{3} \text{ parallelogram } ABCD,$$

and $\therefore$ parabolic area $ABC = \frac{2}{3}$ circumscribing parallelogram.

2. The volume of a solid of revolution may be determined in a similar manner.

Let ABC be a plane curvilinear area by the revolution of which round AB the solid is generated, and let CB be perpendicular to AB. Then if AB $(= h)$ be divided into n equal parts, and the rectangular parallelogram PN be described on MN the r^{th} part, the cylinder generated by the revolution of PN round $MN = \frac{h}{n}\pi \,.\, PM^2 = \frac{h}{n}\pi \,.\, y_r^2$,

and the volume of the solid

$$= \text{limit . sum of all such cylinders}$$

$$= \pi h \,.\, \text{limit}\, \Sigma \,.\, \frac{y_r^2}{n}, \text{ when } n \text{ is infinite.}$$

Ex. 2. To find the volume of a sphere.

Let ABC be a quadrant of the generating circle radius $= h$.

$$\text{then } y^2 = 2hx - x^2,$$

$$y_r^2 = 2h\frac{rh}{n} - \left(\frac{rh}{n}\right)^2 = h^2\left(\frac{2r}{n} - \frac{r^2}{n^2}\right),$$

and therefore volume of hemisphere

$$= \pi h^3 \text{ limit} \left\{\frac{2}{n^2}(1 + 2 + \ldots n) - \frac{1}{n^3}(1^2 + 2^2 + \ldots + n^2)\right\}$$

$$= \pi h^3 \text{ limit} \left\{\frac{2}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) - \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\right\}$$

$$= \pi h^3\left(1 - \frac{1}{3}\right) = \frac{2}{3}\pi h^3,$$

and therefore volume of sphere

$$= \frac{4}{3}\pi h^3 = \frac{2}{3} \,.\, 2h \,.\, \pi h^2 = \frac{2}{3} \text{ circumscribing cylinder.}$$

Ex. 3. Similarly the volume of a cone and paraboloid may be shewn to be $\frac{1}{3}$ and $\frac{1}{2}$ of the circumscribing cylinder respectively.

3. To find the volume of a pyramid.

Let A be the area of the base of the pyramid, and let the perpendicular from the vertex upon the base $= h$. Divide h into n equal parts, and through the r^{th} point of division from the vertex draw a plane parallel to the base. Then the area of the section of the pyramid thus made

$$= A\frac{\left(\frac{rh}{n}\right)^2}{h^2} = A\frac{r^2}{n^2};$$

on this area as a base describe a right prism, whose altitude $= \frac{h}{n}$; then volume of prism

$$= A\frac{r^2}{n^2}\cdot\frac{h}{n} = Ah\frac{r^2}{n^3};$$

and therefore volume of pyramid = limit of sum of all such prisms

$$= Ah \text{ limit } \Sigma\frac{r^2}{n^3} = Ah \text{ limit } \frac{1}{n^3}\,(1^2 + 2^2 + \ldots\ldots + n^2)$$

$$= Ah \text{ limit } \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \tfrac{1}{3}Ah = \tfrac{1}{3}.\text{ base} \times \text{altitude}.$$

Note to Prop. XIII. on Curvature.

4. To find the chords of curvature through the center and focus, and the diameter of curvature, at any point of an ellipse and hyperbola. (Vide Figs. Props. XI. and XII.)

Let Qv, a semi-ordinate to the diameter PCG, cut SP in x, CP in v, and PF, which is perpendicular to the semi-conjugate CD, in u.

Chord of curvature through C

$$= \text{limit}\ \frac{PQ^2}{\text{subtense parallel to } CP}$$

$$= \text{limit}\ \frac{PQ^2}{Pv} = \text{limit}\ \frac{Qv^2}{Pv}$$

$$= \text{limit}\ \frac{CD^2}{CP^2} . vG, \quad \text{since}\ \frac{Qv^2}{Pv \,.\, vG} = \frac{CD^2}{CP^2}$$

$$= \frac{2CD^2}{CP}, \ \text{since } vG \text{ ultimately} = 2CP.$$

Chord of curvature through S

$$= \text{limit}\ \frac{PQ^2}{\text{subtense parallel to } SP} = \text{limit}\ \frac{Qv^2}{Px}$$

$$= \text{limit}\ \frac{Qv^2}{Pv} . \frac{Pv}{Px} = \text{limit}\ \frac{CD^2}{CP^2} . vG . \frac{PC}{PE}$$

$$= \frac{2CD^2}{AC}, \ \text{since } PE = AC.$$

Diameter of curvature

$$= \text{limit}\ \frac{PQ^2}{\text{subtense perpendicular to tangent}}$$

$$= \text{limit}\ \frac{Qv^2}{Pu} = \text{limit}\ \frac{Qv^2}{Pv} . \frac{Pv}{Pu} = \text{limit}\ \frac{CD^2}{CP^2} . vG . \frac{PC}{PF}$$

$$= \frac{2CD^2}{PF}.$$

COR. Let PF cut the axis major in K, then $PF \cdot PK = BC^2$;

$$\text{also } CD \cdot PF = AC \cdot BC,$$

$$\therefore \text{ diameter of curvature} = \frac{2AC^2 \cdot BC^2}{PF^3} = \frac{2AC^2 \cdot BC^2 \cdot PK^3}{BC^6}$$

$$= \frac{8PK^3}{L^2}.$$

5. To find the chord of curvature through the focus, and the diameter of curvature at any point of a parabola. (Vide Fig. Prop. XIII.)

Let QV, a semi-ordinate to the diameter PV, cut SP in X; and the normal PK in U, draw SY perpendicular to the tangent at P; then $PX = PV$, hence

Chord of curvature through S

$$= \text{limit } \frac{PQ^2}{\text{subtense parallel to } SP}$$

$$= \text{limit } \frac{PQ^2}{PX} = \text{limit } \frac{QV^2}{PV}$$

$$= 4SP, \text{ since } QV^2 = 4SP \cdot PV.$$

Diameter of curvature

$$= \text{limit } \frac{PQ^2}{\text{subtense perpendicular to tangent}} = \text{limit } \frac{QV^2}{PU}$$

$$= \frac{4SP \cdot PV}{PU} = 4SP \cdot \frac{SP}{SY}, \text{ by sim}^{\text{r}}\text{. triangles } PVU,\ SPY,$$

$$= \frac{4SP^2}{SY}, \text{ or } = 4\sqrt{\frac{SP^3}{SA}}.$$

Cor. Let PY meet the axis in T, then

$$ST = SP = SK, \quad \therefore \; SY = \tfrac{1}{2} PK;$$

hence diameter of curvature

$$= 4\frac{SP^2}{SY} = 4\frac{SY^4}{SA^2 . SY} = \frac{4}{SA^2} SY^3 = \frac{1}{2SA^2} . PK^3$$

$$= \frac{8PK^3}{L^2}.$$

Note to Prop. VI. Cor. 3.

6. If $SP = r$, and $SY = p$, $PV = \dfrac{2p}{d_r p}$.

(Miller's *Differential Calculus*. Art. 95.)

$$\therefore \; F = \frac{2h^2}{SY^2 . PV} = \frac{2h^2}{p^2 . \dfrac{2p}{d_r p}}$$

$$= \frac{h^2}{p^3} . d_r p.$$

Again, if $r = \dfrac{1}{u}$, $\dfrac{1}{p^2} = (d_\theta u)^2 + u^2$.

(Miller's *Differential Calculus*. Art. 89.)

$$\therefore \; \frac{1}{p^3} d_r p = -\frac{1}{p^3} . d_u p . u^2 = -\frac{1}{p^3} \frac{d_\theta p}{d_\theta u} . u^2$$

$$= (d_\theta u . d_\theta^2 u + u d_\theta u) \frac{u^2}{d_\theta u}$$

$$= u^2 (d_\theta^2 u + u);$$

$$\therefore \; F = h^2 u^2 (d_\theta^2 u + u).$$

Ex. 1. To find the law of force, by which a body may describe the curve $p = \dfrac{ar}{\sqrt{b^2 + r^2}}$, round a center of force in the pole.

$$\frac{1}{p^2} = \frac{b^2}{a^2 r^2} + \frac{1}{a^2},$$

$$\text{and } \therefore \ \frac{1}{p^3} \, d_r p = \frac{b^2}{a^2 r^3};$$

$$\therefore \ F = \frac{h^2}{p^3} \, d_r p = \frac{h^2 b^2}{a^2 r^3} \propto \frac{1}{r^3}.$$

Ex. 2. To find the law of force by which a body may describe a conic section, round a center of force at one extremity of the axis major.

Let S be the center of force at extremity of axis major SA, P any point in the curve, PN perpendicular to SA.

$$\text{Let } SN = x, \quad PN = y,$$

$$\therefore \ y^2 = 2mx + nx^2, \text{ is the equation to the curve.}$$

$$\text{Let } SP = r, \quad PSN = \theta;$$

$$\therefore \ r^2 \sin^2\theta = 2mr \cos\theta + nr^2 \cos^2\theta,$$

$$r = \frac{2m \cos\theta}{\sin^2\theta - n \cos^2\theta} = \frac{2m \cos\theta}{1 - (1+n)\cos^2\theta};$$

$$\therefore \ u = \frac{1}{2m} \left\{ \frac{1}{\cos\theta} - (1+n)\cos\theta \right\},$$

$$d_\theta u = \frac{1}{2m} \left\{ \frac{\sin\theta}{\cos^2\theta} + (1+n)\sin\theta \right\};$$

$$\therefore\ d_\theta^2 u = \frac{1}{2m}\left\{\frac{1}{\cos\theta} + \frac{2\sin^2\theta}{\cos^3\theta} + (1+n)\cos\theta\right\},$$

$$d_\theta^2 u + u = \frac{1}{m}\left(\frac{1}{\cos\theta} + \frac{\sin^2\theta}{\cos^3\theta}\right)$$

$$= \frac{1}{m}\sec^3\theta,$$

$$\therefore\ F = h^2u^2(d_\theta^2 u + u)$$

$$= \frac{h^2}{m}\cdot\frac{\sec^3\theta}{r^2},\ \text{ or } \propto \frac{SP}{SN^3}.$$

Note to Prop. X. Cor. 3.

6. To find the magnitude and position of the axes of the orbit described.

Let $CP = r$, $CPy = a$, $s\left(=\frac{V^2}{2F}\right)$ = space due to velocity of projection, a and b the semi-axes of the orbit described.

$$CD = \sqrt{\tfrac{1}{2}CP\,.\,PV} = \sqrt{2rs},$$

$$\left.\begin{aligned} a^2 + b^2\ (&= CP^2 + CD^2) = r^2 + 2rs \\ ab\ \ (&= CD\,.\,PF) = \sqrt{2rs}\,.\,r\sin a \end{aligned}\right\},$$

from which two equations a and b, and therefore e, the eccentricity, may be determined.

Also if θ be the inclination of axis major to CP,

$$r = \frac{b}{\sqrt{1 - e^2\cos^2\theta}},$$

$$\therefore\ \cos\theta = \frac{1}{e}\sqrt{1 - \frac{b^2}{r^2}},$$

which may therefore be determined.

NOTE TO PROP. XIII. COR.

7. To find the magnitude and position of the axes of the orbit described.

Let $SP = r$, $\angle SPY = \alpha$, draw SY, HZ perpendicular to YPZ; and let a, b, L be the semiaxes and latus rectum of the orbit;

$$\text{then } PL . PH = PK . PS,$$

$$\text{or } (r \sim s) . (2a \mp r) = s . r,$$

$$\therefore 2a(r \sim s) - r^2 = 0,$$

$$\therefore a = \frac{r^2}{2(r \sim s)},$$

$$b = \sqrt{SY . HZ} = \sqrt{SP \sin\alpha . HP . \sin\alpha}$$

$$= \frac{r . s^{\frac{1}{2}} . \sin\alpha}{(r \sim s)^{\frac{1}{2}}};$$

$$\therefore L = \frac{2b^2}{a}$$

$$= 4s . \sin^2\alpha.$$

Again, $$YZ = SH . \sin ASY,$$

$$\text{or } (SP + HP) \cos\alpha = 2e . AC \sin ASY,$$

$$\therefore \sin ASY = \frac{1}{e}\cos\alpha,$$

which equation, since $e = \sqrt{1 \mp \frac{b^2}{a^2}}$ is known, determines the position of the axis major.

ON ANGULAR VELOCITY.

8. The angular velocity of a body, moving in an orbit round a center of force S, (fig. page 28.) is measured by the angle uniformly described by SP round S in $1''$, in the same manner as linear velocity is measured by the line uniformly described in $1''$. If the angular motion of SP be not uniform, the angular velocity at any point is measured by the angle, which would be described in $1''$, if the angular motion of SP were to continue uniform for that time. Hence if the angular motion be not uniform, and PSQ be the angle described in T'' after leaving P, the angular velocity

$$= \text{limit } \frac{\text{angle } PSQ}{T},$$

for this is the angle which would be described in $1''$, if the angular motion at P were to continue uniform for that time.

Prop. *If a body be moving in any orbit round a center of force S, the angular velocity at any point P*

$$= \frac{h}{SP^2}.$$

Let PSQ be the angle described in T'', with center S and radius SQ, describe a circular arc cutting SP in T, and draw SY perpendicular to the tangent at P; then the triangle PTQ may be considered as ultimately rectilinear, and similar to SYP, hence

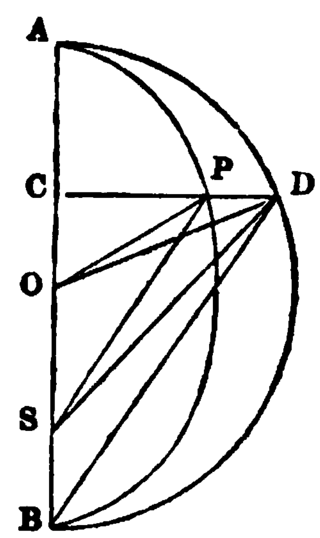
$$\angle^{r} \text{ vel. at } P = \text{limit } \frac{\angle PSQ}{T} = \text{limit } \frac{QT}{SQ \,.\, T}$$

$$= \text{limit } \frac{PQ \,.\, SY}{SP^2 \,.\, T}, \text{ since limit } \frac{QT}{PQ} = \frac{SY}{SP}$$

$$= \frac{SY \,.\, \text{vel. at } P}{SP^2}, \text{ since limit } \frac{PQ}{T} = \text{vel. at } P$$

$$= \frac{h}{SP^2}, \text{ (Prop. I. Cor. 3).}$$

9. *Force varying as (distance)$^{-2}$. To find the time of motion and the velocity acquired by a body falling through a given space from rest.* (PROPS. XXXIII. and XXXVI.)



Let S be the center of force, A the point from which the body begins to fall;

$$\frac{\mu}{SP^2} = \text{force at distance } SP.$$

Let APB be a semi-ellipse, focus S and axis major ASB; ADB a semi-circle, whose diameter is ASB; and suppose a body revolving in the ellipse round the focus S to come to P; bisect AB in O, draw DPC perpendicular to AB, and join OP, OD.

Then the time through $AP \propto$ area $ASP \propto$ area ASD, and this being true for all values of the axis minor will be true when it is diminished without limit, in which case the ellipse coincides with the axis major and the point P with C, or the body is moving in the straight line AC; the point B also coincides with S, since $AS \cdot SB = (\frac{1}{2}\text{ axis minor})^2$; and since space due to velocity at $A = \frac{1}{4}$ chord of curvature at A through $S = \frac{1}{4}$ latus rectum $= \dfrac{(\text{axis minor})^2}{4AB} = 0$, the body begins to move from rest at A.

Hence time from rest through $AC \propto$ area ABD,

$$\therefore \frac{\text{time rhrough } AC}{\text{time through } AB \, (= \frac{1}{2} \text{periodic time in ellipse})} = \frac{\text{area } ABD}{\text{semi-circle } ABD};$$

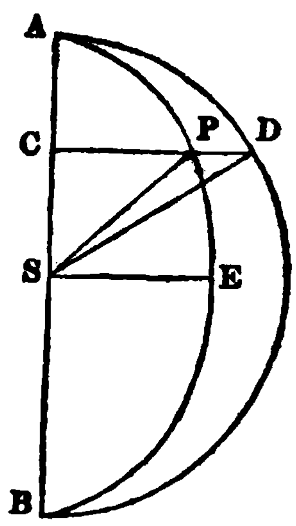
$$\therefore \text{ time through } AC = \frac{\pi \cdot AO^{\frac{3}{2}}}{\sqrt{\mu}} \cdot \frac{\frac{1}{2} AO \cdot (AD + CD)}{\frac{1}{2}\pi \cdot AO^2}$$

$$= \sqrt{\frac{AS}{2\mu}} \cdot (AD + CD).$$

Again, velocity at $P = \sqrt{\frac{\mu}{AO} \cdot \frac{HP}{SP}}$ (Prop. XVI.) and when the ellipse coincides with the axis major,

$$\text{velocity at } C = \sqrt{\frac{2\mu}{AS} \cdot \frac{AB - BC}{BC}} = \sqrt{\frac{2\mu}{AS} \cdot \frac{AC}{SC}}.$$

10. *Force varies as distance. To find the time of motion and the velocity acquired by a body in falling through a given space from rest.* (Prop. XXXVIII.)



Let S be the center of force, A the place from which the body begins to fall: on $AB = 2AS$ describe a semi-ellipse APB, and a semi-circle ADB, and let a body moving in the ellipse come to P. Draw DPC perpendicular to AB, and join SP, SD.

Then time through $AP \propto$ area $ASP \propto$ area ASD, and this being true, whatever be the axis minor of the ellipse, will be true when it is diminished without limit, in which case the body will be at C, having fallen from rest at A,

$\therefore$ time through $AC \propto$ area ASD

$$\therefore \frac{\text{time through } AC}{\text{time through } AS \left(= \frac{1}{4} \text{ periodic time in a circle}\right)}$$

$$= \frac{\text{sector } ASD}{\frac{1}{4} \text{ area of a circle}};$$

$$\therefore \text{ time through } AC = \frac{\pi}{2\sqrt{\mu}} \cdot \frac{\frac{1}{2} AS \cdot AD}{\frac{1}{4}\pi AS^2}$$

$$= \frac{AD}{AS\sqrt{\mu}}.$$

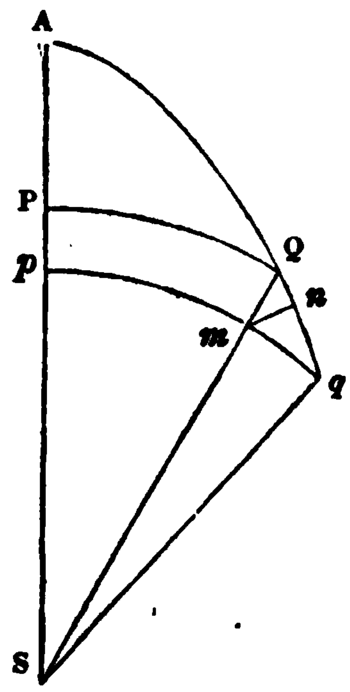
Again, let SE be the semi-axis minor,

then vel. at P = semi-conjugate at $P.\sqrt{\mu}$. (Prop. x. Cor. 1).

$$= \sqrt{AS^2 + SE^2 - SP^2}.\sqrt{\mu},$$

$$\therefore \text{ vel. at } C = \sqrt{AS^2 - SC^2}.\sqrt{\mu}$$
$$= CD\sqrt{\mu}.$$

11. *If the velocities of two bodies, one of which is falling directly towards a center of force, and the other describing a curve about that center, be equal at any equal distances, they will always be equal at equal distances.* (Prop. XL.)



Let S be the center of force, and let one of the bodies be moving in the straight line APS and the other in the curve AQq; with radii SQ, Sq describe the circular arcs QP, qp: let SQ cut pq in m, and draw mn perpendicular to Qq; and suppose the velocities of the bodies at P and Q to be equal.

Since the centripetal forces at P and Q are equal, Pp, Qm may be taken to represent them: Pp is wholly effective in accelerating P, but the effective part of Qm is Qn, nm being wholly employed in retaining the body in the curve. Also since the velocities at P and Q are equal, the times of describing Pp and Qq, when the spaces are diminished indefinitely, are proportional to Pp and Qq; hence

$$\text{force at } P : \text{force at } Q = Pp : Qn$$

$$\text{and time through } Pp : \text{time through } Qq = Pp : Qq;$$

$$\therefore \text{ velocity acquired at } p : \text{velocity acquired at } q$$
$$= Pp^2 : Qn.Qq = Qm^2 : Qn.Qq$$
$$= 1 : 1,$$

and the same may be shewn at all corresponding points equally distant from S, therefore, *If the velocities, &c.*

ERRATA.

Page 1, in the last line but one, after "then" insert "the ratio of."

— 16 line 2, *for* RQr *read* RQq.

— 29 — 10, *for* chords *read* chord.

— 50 — 8, *for* NR *read* NQ.

OUTLINES

OF

A NEW THEORY

OF

ROTATORY MOTION,

TRANSLATED FROM THE FRENCH OF POINSOT, Louis

WITH

EXPLANATORY NOTES,

BY

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ADVERTISEMENT.

The following pages contain a version of the Extract which M. Poinsot has published of the Memoir presented by him to the French Institute on the 19th of May 1834.

The object of the Translator is to call the attention of those engaged in Mathematical pursuits in this country to a subject which has acquired so great an interest for the Mathematicians of the Continent from the animated discussions respecting it which have arisen out of the above Memoir.

With a view to facilitate the progress of the Reader, but more especially in the hope of rendering this publication useful to Students in the University, demonstrations of the fundamental propositions have been subjoined in the form of notes, and an Appendix containing demonstrations of the leading principles assumed, viz. the existence of an Instantaneous Axis, and of three Principal Axes, and the Conservation of Couples, has been added.

The Translator has been informed by a distinguished Member of the Institute that the Memoir will appear entire in the fourth of a succession of volumes to be published by that Society in the course of next year.

ERRATA.

PAGE	LINE	
2	5 from bottom	*for* "rotation" *read* "rotatory motion."
4	last line	*for* "rotation" *read* "rotatory motion."
14	Note (6)	*after* "bb'" *insert* "(fig. 7.)".
20	18 from top	*after* "moveable cone" *insert* "whose vertex is O."
24	22	*for* "same *furrow*" *read* "equal *furrows*."

THEORY

OF

ROTATORY MOTION.

INTRODUCTORY REMARKS.

THE following enquiry in Dynamical Science is one which has most frequently occupied my attention, and forms one of the subjects which, if I may so say, I have been most anxious thoroughly to understand.

Every one can form for himself a clear idea of the motion of a point, that is to say, of a corpuscle supposed to be infinitely small, which gives us in some degree the notion of a mathematical point. For we have only to figure to ourselves the line, straight or curved, which the point may describe, and the velocity with which it moves in this line. But if we have to consider the motion of a body of sensible magnitude and definite shape, it must be allowed that the idea which we form of it is very obscure.

At first indeed, the idea appears to become clearer by resolving itself into two others. For

if we confine ourselves to the consideration of one particular point in the body, we may on the one hand follow the motion of this point, which can only describe a certain line in space, and on the other the motion of the body, which can only turn at the same time on this point as about a fixed axis. But this second motion, namely, that of a body moveable about a point, round which it is at liberty to turn in every direction, is one of which we have but a confused notion.

Not but that, by referring the different points in the body to planes or objects fixed in space, we can find *differential equations*, as they are called, of this motion, which, in the simple case of a body acted on by no external force, we have even been able to *integrate*, or at least to reduce to *quadratures*. Euler and d'Alembert (nearly at the same time and by different methods) were the first to solve this important and difficult problem: and afterwards, as is known, the illustrious Lagrange undertook to investigate anew this famous question, and to develop it in his own manner; I mean, by a series of analytical formulæ and transformations, remarkable for their symmetry and elegance. But it must be allowed that in all these solutions we see nothing but calculations, without having any clear idea of the rotation of the body. We may be able, by means of calculations, more or less long and complicated, to determine the place of the body at the end of a given time; but we do not see at all how it arrives there. We are totally unable to keep

it in view and to follow it, as we might wish, with our eyes, during the whole course of its rotation.

Therefore to furnish a clear idea of this Rotatory Motion, hitherto unrepresented, has been the object of my endeavours*.

The result is an entirely new solution of the problem of the rotatory motion of a body, acted on by no force, whether it turns freely on its centre of gravity, or on any other fixed point about which it is constrained to move: a genuine solution, inasmuch as it is palpable, and enables us to follow the motion of the body as clearly as the motion of a point. And if we would pass from this geometrical representation to calculation, in order to measure all the different properties or affections of this motion, the formulæ requisite for the purpose are direct and simple, each of

* It has always appeared to the Translator, that Geometrical illustrations, whenever they can be obtained, are of very great use in enabling us to form clear and correct ideas of the meaning of analytical formulæ and operations. It may be sufficient to advert to the connexion between the singular solution of a differential equation, and the envelope of series of curves described after a given law, or between the first integrals of a partial differential equation of the second order, and the characteristics of the curved surface to which it belongs. In quadratures particularly, a large class of integrals may be made to depend upon elliptic arcs; and it may be remarked here, that a distinguishing feature of M. Poinsot's theory is the explanation of all the complications of rotatory motion by a reference to the properties of an ellipsoid, with which Analytical Geometry has made us familiar: while the calculations for determining the actual position of the body resolve themselves at once into the form of the elliptic transcendents here mentioned.

them expressing a dynamical theorem of which we have a clear idea, and which proceeds at once to its object. My analysis of the question therefore offers in addition this advantage, that every thing therein is expressed and developed in terms of the immediate conditions of the problem, without the intervention of those coordinates and angles which are foreign to the question, and which take their rise only in the indirect method employed to discuss it. For we may remark generally of our mathematical researches, that these auxiliary quantities, these long and difficult calculations into which we are often drawn, are almost always proofs that we have not in the beginning considered the objects themselves so thoroughly and directly as their nature requires, since all is abridged and simplified, as soon as we place ourselves in a right point of view.

I thought then that a solution so simple, and so well calculated to throw new light on the most difficult questions of Dynamics, might further the real advancement of science, and was therefore worthy the attention of Geometers: and this consideration led me to compose the Memoir, which I had the honour this day to present to the Institute.

I divide it into three principal sections. In the first I consider the actual motion of the body, and the forces which would be capable of producing it. In the second, I give the solution of the problem of the rotatory motion of a free body;

and in the third, I develope the calculations which relate to this solution.

But to give a more precise idea of this work, I shall briefly lay down the first principles of the new theory, and afterwards go hastily over the principal theorems which are the object and result of it.

SECTION I.

ON THE ACTUAL MOTION OF BODIES, AND ON THE FORCES CAPABLE OF PRODUCING ANY GIVEN MOTION.

Signification of the terms, Simple Rotatory Motion, and Angular Velocity.

THE only rotatory motion of which we have a clear idea, is that of a body which turns on an immoveable axis. For we see plainly all the circumferences of the circles which the different points of the body describe about this axis, and which they can really describe at the same time, without changing in any respect their relative position, or what we may denominate the form of the body.

We have an equally clear idea of the quantity or measure of this rotatory motion; for since all the points in it describe similar arcs in the same time, the ratio of the velocity of a point to the radius of the circle which it describes is the same for all points, and it is this constant ratio which forms the measure, or, as it is called, the *angular velocity* of rotation. (1)

(1) Let OPp (fig. 1) be the axis of rotation, R, r, any two points describing the circles RR', rr', which must lie in planes perpendicular to Op, and therefore parallel to each other.

Let RR', rr', be arcs described uniformly in the same time (t).

Draw PS parallel to pr, and when pr comes into the position pr', let PS' be the corresponding position of PS.

Then $\angle SPS' = \angle rpr'$. (Euc. XI. 10.)

But since the body is rigid,

$$SR = S'R',$$

and therefore, $\angle RPR' = \angle SPS' = \angle rpr'$.

Hence every point in the body describes round Op in the time (t) an angle $= RPR'$.

Let (ω) be the angle described in $1''$ by every point,

$$\text{then } \angle RPR' = t\omega,$$

and the actual velocity (v) of $R = \dfrac{RR'}{t}$

$$= \frac{PR \,.\, \angle RPR'}{t} = PR \,.\, \omega,$$

and $\dfrac{v}{PR} = \omega$, which is the same for every point, is called the *Angular Velocity* of the body, or the *Velocity of Rotation.*

Composition of Rotatory Motion.

From these simple notions, and from the primary elements of Geometry, we may conclude that if, from the influence of any two separate causes, a body tended to turn at the same time round

the two sides of a parallelogram, with two angular velocities respectively proportional to the lengths of these sides, the body would turn round the diagonal, with an angular velocity proportional to the length of this diagonal. (2)

(2) Let the straight lines Oa, Ob, (fig. 2.) lying in the plane of the paper intersect each other in O, and suppose two impulses to act simultaneously upon a body, one of which would cause it to turn about Oa with an angular velocity $= e \,.\, OA$, and the other to turn about Ob with an angular velocity $= e \,.\, OB$. Draw AP, BP, parallel to OB, OA, and PM, PN perpendicular to OA, OB; and let QPQ' be the circle which the point P would describe, if the first impulse were communicated singly, about Oa,

RPR' ....
.... second ....
.... Ob.

Then the planes in which these circles lie will both of them be perpendicular to the plane bOa passing through the axes, and therefore their intersection pPp' will be perpendicular to this plane, and therefore to each of the lines PM, PN, which are the radii of the circles, and will therefore be a tangent to both circles at P.

The first impulse therefore, would make P tend to move in the direction Pp with a velocity $= MP \times$ (angular velocity round Oa)

$$= e \,.\, OA \,.\, MP = e \,.\, OA \,.\, OB \sin BOA,$$

and the second in the direction Pp' with a velocity $= NP \times$ (angular velocity round Ob)

$$= e \,.\, OB \,.\, NP = e \,.\, OB \,.\, OA \sin BOA,$$

that is, in a direction exactly opposite, with the same velocity. If therefore these impulses were communicated

together, the point P would remain at rest; and the same may evidently be proved of every other point in the line OP, which would therefore remain entirely at rest in consequence of the two impulses.

Draw AK, AL perpendicular to OB, OP. Then it is evident that the first impulse would not cause any tendency to motion in the point A, and the second would cause it to move in the direction of the tangent to a circle whose radius is AK, and whose plane is perpendicular to Ob, that is, in a direction perpendicular to the plane bOa, with a velocity $= e \,.\, OB \,.\, AK$. But in consequence of the two impulses, the line OP remains fixed in space, and since the body is rigid, the distance AL is invariable, and the direction of A's motion coincides with the tangent to a circle whose radius is AL, and whose plane is perpendicular to OP. The point A will therefore describe about L the circle SAS', with an angular velocity $= \dfrac{e \,.\, OB \,.\, AK}{AL} = e \,.\, OP$, (by similar triangles OPN, PAL.) And in the same manner, it may be shewn that every point in OA will describe a circle, in a plane perpendicular to OP, with an angular velocity $= e \,.\, OP$. Hence the whole line OA, and consequently the rigid body in which it lies, will turn about OP with an angular velocity $= e \,.\, OP$.

The same reasoning clearly holds if an interval occur between the communication of the impulses.

The velocities of the highest points Q' and R of the circles described by a point lying within the $\angle\, bOa$, are here supposed to be in the same direction, when resolved perpendicularly to a plane bisecting $\angle\, bOa$; and the rotatory motions are consequently said to be in the same direction.

If they are in opposite directions, we must take $OB' = OB$ (fig. 3.) on the other side of OA and complete

the parallelogram, when OP' may be shewn as before to be the new axis, and $e \cdot OP'$ the angular velocity of the body about it.

Whence it follows that the rotatory motions about different axes which intersect in any point are compounded in precisely the same manner as simple forces applied at the point.

And this similarity of composition is not confined to rotatory motions about axes which intersect; but what is very remarkable, it extends to rotatory motions about axes situated any how in space.

Thus rotatory motions about two parallel axes are compounded into a single one equal to their sum, about an axis parallel to them, which divides their distance in the inverse ratio of the component rotatory motions. (3)

(3) Let the axes aa', bb' (fig. 4.) be as before in the plane of the paper. Draw AB perpendicular to them, and let

$$\text{angular velocity round } aa' = e \cdot BP,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad bb' = e \cdot AP,$$

then as before,

Velocity of P upwards, in a direction perpendicular to the plane of the paper, due to the rotatory motion round aa', $= AP \times$ (angular velocity round aa')

$$= e \cdot BP \cdot AP.$$

Velocity downwards, due to the rotatory motion round bb',

$$= e \cdot AP \cdot BP.$$

And P, and similarly every point in pp', remains at rest, and therefore pp' becomes the new axis of rotation.

And the velocity of A perpendicular to AP, which is that due to the rotatory motion round bb', $= e \,.\, AP \,.\, AB$; therefore velocity of A round P, or velocity of the body's rotation round pp',

$$= \frac{e \,.\, AP \,.\, AB}{AP} = e \,.\, AB = e \,.\, (BP + AP),$$

$$= (\text{velocity of rotation round } aa')$$
$$+ (\text{velocity of rotation round } bb').$$

What is meant by saying that the rotatory motions are in the same direction is evident from the last note: that is, that the motions of Q' and R are in the same direction.

If these are in opposite directions, the resultant rotatory motion is equal to their difference, and the position of the axes is determined by the same laws as that of the resultant of two parallel forces acting in opposite directions. (4)

(4) If the rotatory motions are in opposite directions, the motion of r', (fig. 5.) the highest point of the circle described by a point P' round bb', will be in an opposite direction to that of Q', which we may suppose to be the same as in the last note.

Suppose the angular velocities to be unequal, and let that round aa' be the greater. In BA produced take a point P, such that

angular vel. round aa', which we may call ω_a, $= e \,.\, PB$,
.... bb', ... ω_b, $= e \,.\, PA$,

then it is evident that the motion of R' will be parallel to that of r', and the motion of P round bb' will be in the direction RPR', while the motion round aa' is in the direction $Q'PQ$; and therefore as before,

velocity of P downwards $= \omega_a \,.\, PA = e \,.\, PB \,.\, PA$,
.... upwards $= \omega_b \,.\, PB = e \,.\, PA \,.\, PB$;

therefore pp' is the new axis of rotation, and the velocity of rotation round pp', or ω_p,

$$=\frac{e\,.\,PB\,.\,AB}{PB}=e\,.\,AB=ePB-ePA=\omega_a-\omega_b.$$

We may therefore conclude that the resultant of two rotatory motions ω_a, ω_b, about parallel axes aa', bb' is a rotatory motion about an axis parallel to them in the same plane, which $=\omega_a\pm\omega_b$, according as the component rotatory motions are in the same or opposite directions; the distance of the axis from bb' being equal to BP

$$=BA\,.\,\frac{e\,.\,BP}{eBP\pm eAP}=BA\,.\,\frac{\omega_a}{\omega_a\pm\omega_b},$$

BA being measured in a positive direction.

If the motions are in opposite directions and $\omega_b>\omega_a$, BP will be negative, and P will lie on the other side of B.

If $\omega_a=\omega_b$, BP will be infinite, and every point will describe a circle of infinite radius. This case is considered in the next note. (*See Pritchard's Couples, Appendix*, Prop. C.)

If these two parallel and opposite rotatory motions are equal, they can never be reduced to a single one. They form in that case what may be called a *Couple of Rotatory Motions:* a rotatory motion *sui generis*, and which can never be reduced to a simple rotatory motion about any axis whatever. And in fact it is easy to see that the result of such a couple would be to give the body a simple motion of translation in space, in a direction perpendicular to the plane of the couple, and measured by its *moment*, that is, by the product of one of

the rotatory motions and the distance between the parallel axes. (5)

(5) Let the plane of the paper be *perpendicular* to the parallel axes aa', bb', (fig. 6.) which meet it in the points A and B; and suppose two impulses to be communicated to the body which, separately, would generate angular velocities, each $= \omega_a$, round them in opposite directions. Take $AB = a$. Then it is clear that if for $1''$ the motion round aa' were alone communicated, B would describe the arc $BB'' = a\omega_a$, and if the motion round aa' were then to cease and that round bb' to commence, A would describe an arc AA'' also $= a \cdot \omega_a$. At this time therefore the position of the distance between the axes ($A''B''$) is parallel to the initial position (AB) and the points of intersection (A) and (B) of the axes with the paper have moved in the direction BA through $A'A'' = a \cdot \text{versin}\, \omega_a$. If we suppose the motions to be communicated in reversed order we shall have the new position of the distance ($A'''B''$) still parallel to AB, while the points of intersection will have moved in the direction AB through $A'A''' = a \cdot \text{versin}\, \omega_a$. And it is clear that the effects will be similar however small the intervals may be taken. If therefore the impulses be communicated together, the direction of the distance at any instant will be parallel to AB, and consequently the motion round bb', which tends to make A move in the direction AB, will tend (since AB may be considered as an incompressible rod) to make B move in the same direction, while an equal and contrary effect will be produced directly on B by the motion round aa'. Hence every point in AB and consequently every point in the body will move in a direction perpendicular to AB.

And if P be any point in AB,

$$P\text{'s velocity} = AP \cdot \omega_a + BP \cdot \omega_a$$
$$= a \cdot \omega_a; \quad \text{which is the moment of the couple.}$$

A *Couple of Rotatory Motions* is therefore equivalent to a single force, applied at the centre of gravity of the body, in a direction perpendicular to the plane of the couple, and equal to the product of its moment by the mass of the body.

These couples may be transformed into others equivalent to them, may be turned about or removed at pleasure in their own planes or into planes parallel to their own, without any change being produced in the motion of the body. Their composition and decomposition follow exactly the same law with those of ordinary couples, and we may apply to them without exception the corresponding theorems. (6)

(6) If aa', bb', *in* the plane of the paper be the axes of a couple whose moment is $a \,.\, \omega_a$, it is evident that the same motion will be produced in the point P, perpendicular to the plane, by a couple of rotatory motions round cc', dd', whose distance $CD = b$, with angular velocities each $= \omega_b$, if $a \,.\, \omega_a = b \,.\, \omega_b$. (*See Pritchard*, Prop. IV.)

Also if the axes be turned in the same plane into the positions ee', ff', parallel to each other the effect on P is the same. (v. *Pritchard*, Prop. III.)

And what is here true of P is true of every point in the body.

The propositions corresponding to (*Pritchard*, Prop. II. and Cor.) are self-evident.

The effect of any number of couples in the same or parallel planes is clearly equal to their sum, the angular velocities being taken with their proper signs.

If two couples lie in planes inclined to each other they may be transformed and removed, until their

distances are equal and coincide with the intersection of the planes and with each other, (v. *Pritchard*, Prop. VI.) when the motions of translation which they produce in the point P, taken as before, may be compounded as in Elementary Dynamics. And the same will apply to the parallel motions of every other point in the body.

From the parallelogram of *Simple Rotatory Motions*, and the parallelogram of *Couples of Rotatory Motions* results the composition of any number whatever of rotatory motions, situated any how in space, and this general composition perfectly resembles the general composition of forces. (7)

(7) If $\angle\, bOa = 90^{0}$ (fig. 8.), we have from note (2) the rotatory motion round Op with an angular velocity $(\omega_p) = e \,.\, OP$, equivalent to

rotatory motion round Oa with angular vel. $(\omega_a) = e\,.\,Oa$,
.... Ob .... $(\omega_b) = e\,.\,Ob$;

$$\therefore\ (\omega_p)^2 = (\omega_a)^2 + (\omega_b)^2,$$

$$\text{and } \begin{array}{l} \omega_a = \omega_p \cos a, \\ \omega_b = \omega_p \sin a, \end{array} \quad \text{where } a = pOa = \tan^{-1}\frac{\omega_b}{\omega_a},$$

and similarly if Op be the diagonal of a parallelopiped, and a, β, γ, the angles which it makes with the three edges passing through O, the rotatory motion about it may be resolved into three rotatory motions about these axes with angular velocities,

$$\omega_p \cos a, \quad \omega_p \cos \beta, \quad \omega_p \cos \gamma, \text{ respectively.}$$

We may therefore resolve any number of rotatory motions about axes passing through a point, into others about three rectangular axes passing through that point,

which we may call the origin, and taking the sums about each axis, compound them again for a general resultant.

If any one of the axes do not pass through the origin we must suppose two opposite rotatory motions, each equal to the motion about it, to be impressed upon the body, about an axis parallel to it through the origin: we shall then have a motion about the latter axis, and also a couple of rotatory motions to consider, in addition to any supposed to exist previously in the body. And any number of couples of rotatory motions impressed on the body may be resolved into equivalents in the three co-ordinate planes, and a resultant obtained as for the simple rotatory motions.

Thus, in the same way that any number of forces whatever may always be reduced to a single one passing through a given point and a single couple, any number of rotatory motions about different axes situated any how in space, may always be reduced to a single rotatory motion about an axis passing through a point, selected at pleasure, and a single couple of two equal and opposite rotatory motions about axes parallel to one another. And if, as in the case of forces, we wish to reduce them so as to leave nothing arbitrary, we may always assign a position for the point above-mentioned, such that the plane of the *couple* shall be perpendicular to the axis of the resultant rotatory motion, which may in this case be denominated the *Central Axis* of the couples of rotatory motions*.

Since a couple of rotatory motions is equivalent to a simple motion of translation of the body in

* See *Pritchard*, Prop. VII.

the direction of the *axis* of the couple, the above analysis ultimately reduces the whole motion to a rotatory motion about a determinate axis, and a simultaneous motion of translation in the direction of this axis. Which is, as we shall see farther on, the most general motion that a body can have in absolute space.

Rotatory Motion about a Point.

We now proceed to give an idea of the motion of a body about a point on which it appears to turn in every direction.

The motion of a body which turns on an immoveable axis being the only one of which we have a clear idea, it is to our notions thereof that we must endeavour to reduce the motion of a body turning any how about a fixed point.

Now it is shewn that this motion, whatever it may be, if considered only for a single instant, is none other than a simple rotatory motion about a certain axis passing through the fixed point, whose direction remains immoveable during this instant*.

Hence it is concluded that in the following instant it is likewise a simple motion of rotation, but about another axis; and so on, from one instant to another, in such wise that the motion of the body may be considered as a succession of simple rotatory motions, of each of which the

* See Appendix.

mind has a distinct idea. It is in the same way that, in order to form a notion of the motion of a point in a curved line, we represent this point as describing successively the sides of an infinitesimal polygon inscribed in this curve. And the same is true of the instantaneous axis of rotation in the motion of a body, as of the tangent to a curve in the motion of a point which describes it.

Sensible Illustration of this Rotatory Motion.

Though the preceding analysis is exact, and I have laboured to render it clear, it seems to me that our idea of a body, turning about an axis which is perpetually changing, is still rather obscure. It is therefore desirable to render it clearer, and to present to the mind a distinct and sensible image of it.

Now I demonstrate in the most simple manner that, *the rotatory motion of a body about an axis which incessantly varies its position round a fixed point, is identical with the motion of a certain cone, whose vertex coincides with this point, and which rolls, without sliding, on the surface of a fixed cone having the same vertex.*

I mean to say that the moveable cone, supposed to have a rigid connection with the body, which it carries along with it, if made to roll on the other cone, which is fixed in absolute space, will cause the body to perform the precise motion with which it is supposed to be endued.

The line of contact of the two cones will be at every instant the axis about which the body turns for this instant, or as it is called, the *instantaneous axis;* whence we perceive that this axis is moveable at the same time in the body and in absolute space, describing in space the surface of the fixed cone, and in the interior of the body the surface of the moveable cone of which we have just spoken. (8)

(8) The most general motion of which a rigid body is capable about a fixed point O (fig. 9.) is manifestly such, that any point P in it, taken at pleasure, may be made to describe any given curve of double curvature lying on the surface of a sphere, of which O is the centre, and OP the radius, with a velocity varying as any function of its position.

The normal planes at successive points of such a curve of double curvature intersect in a conical surface*, whose vertex is O. Suppose this surface to be disconnected from the body; and let the body be rigidly connected with a plane which rolls upon the cone, both of them being in the interior of the body and perfectly rough. Let OR be the line of contact and $\angle POR=\alpha$, which will be the inclination of the osculating plane† at P to the rolling plane.

If now an angular velocity (ω) be communicated to the body about OR it may be resolved into two, one about an axis perpendicular to OP in the plane POR, which will cause P to move in the osculating plane, with an angular velocity $= \omega \sin \alpha$, and the other about OP, which being resolved again, part of it about an axis OQ, perpendicular to the rolling plane and therefore in the normal plane POR, will be destroyed

* Hymers' *Analytical Geometry*, Art. 64. † Ibid. Art. 63.

by friction, leaving finally an angular velocity about $OR = \omega \cos \alpha \sin \alpha$.

Let t be the small time which elapses before the plane comes into contact with the next axis of rotation, which will manifestly depend on the form of the cone; and suppose the *whole system* to turn about OR in an opposite direction with an angular velocity β. Then $t.(\omega \sin \alpha \cos \alpha - \beta)$ will be the angle described by the rolling plane in space, and $t\omega \sin \alpha \cos \alpha$ in the system.

Now the inclination of the new position of the osculating plane to the rolling plane, that is, the new value of α, depends on the latter only, while the new position of the osculating plane in space depends on the former. Hence the *relation* between the alteration of P's velocity in the osculating plane and the change in the position of this plane, may be varied arbitrarily by varying β. But we may evidently give what value we please to β at any point, by introducing a moveable cone, instead of a moveable plane, to which the body is rigidly attached, and supposing the system absolutely at rest; for the rolling plane revolving backwards in space would constantly touch such a cone. So that, conversely, we may assign such forms to the cones, and such a function of the position for the velocity of the moveable cone, as to make P describe any curve whatever; with a velocity depending in any manner on the position.

And this, I believe, is the greatest degree of clearness of which an idea so complicated and obscure, as that of the motion of a body which turns any how about a fixed centre, is susceptible. There is no motion of this nature which cannot be produced exactly, by making a certain cone roll on a fixed cone having the same vertex; so that if we figure to ourselves all the possible cones

which can be made to roll in this manner upon one another, we have a faithful illustration of all the possible motions of a body round a point, on which it is at liberty to turn in every direction.

Also, if the rotatory motion to be considered were discontinuous, that is, if the axis of rotation, instead of changing its position by insensible degrees, were to leap abruptly from one position to the next through a finite angle, we could imitate equally well the motion of the body, by taking, instead of two cones, two pyramids having the same vertex, and causing one to roll on the other, so that the moveable pyramid turning on their common edge should bring into contact all its different faces, one after the other, with the faces respectively equal to them of the fixed pyramid.

If the motion of the body is given, it is clear that the two cones or pyramids are also given, as well as the velocity of rotation round the line of contact, and consequently the velocity with which the instantaneous axis traces at the same time the two surfaces. And reciprocally, if of these different quantities, which come under our consideration in discussing the motion, any three are given, we may say that the fourth is also, and that the motion of the body is thoroughly determined.

Thus the Earth turns in one day on its axis, while this axis describes in an opposite direction a right cone* about the axis of the ecliptic, with

* See Appendix.

a velocity measured by the retrograde motion (*Precession*) of the Equinoxes, and which amounts to about 50″ in a year; we may therefore determine in the case of the Earth the cone which, rolling on the former in the interior of the globe, would generate in the Earth a motion corresponding precisely to that which we observe in it. And it is easy to see that on the Earth the circumference of the circle which is the base of this little moveable cone, is to that of the base of the fixed cone, as a day to the period of a complete revolution of the equinoxes: which gives scarcely six feet for the little circumference which the instantaneous pole of rotation of the Earth describes every day on its surface. (This is on the hypothesis of an uniform diurnal Precession.) (9)

(9) Let P (fig. 10.) be the pole of the Earth, K the point where a line perpendicular to the plane of the ecliptic at the Earth's centre meets the surface. Then we may consider the centre of gravity as a fixed centre of rotation; and consequently PQR, the fixed circle in space which the pole describes, will have its circumference equal to $2\pi \times$ radius of Earth $\times \sin(PK$ = Earth's obliquity), and the little circle which rolls upon it, and which is the path of the instantaneous pole on the Earth's surface, will have its circumference equal to the portion of PQR which is described in one day, that is

$$= \frac{\dfrac{1}{365}}{\dfrac{360 \times 60 \times 60}{50}} \times \text{radius of } \oplus \times \sin 23^{0}.28'$$

$= 5\frac{1}{2}$ feet nearly.

And every point in this little circle becomes successively in the course of a day the instantaneous pole of rotation. Hence we may find at any instant the actual position, both in space and on the Earth itself, of the instantaneous pole.

The most general Motion that a Body can have in Absolute Space considered.

From the simple idea of a mere motion of translation, which carries forward at every instant all the equal molecules of the body through small equal and parallel lines in space, and from the simple idea of the rotation of the body about an axis, which remains immoveable during this instant, results the complex idea of the most general motion of which a body is capable in absolute space. Nothing is more clear than this resolution of any kind of motion into, two others which we can conceive perfectly, and which we may consider separately, since they are such that, if at every instant they were executed one after the other, every point in the body would be brought to the same place at which it arrives, by its natural motion, at the end of the instant of which we are speaking.

But we may have a curiosity to form for ourselves a notion of the real and single motion with which the body is endued, for the purpose of seeing, in some degree, the nature of the simultaneous curves which the different points describe, and which

they can really describe at the same time without causing any change in the form of the body.

Now since a motion of translation may always be considered as a *Couple* of equal and opposite rotatory motions, it follows that the motion of a body, whatever it may be, can always be reduced to a simple rotatory motion about an axis passing through a point selected arbitrarily in space, and a certain couple of rotatory motions, whose plane will be in general inclined to this axis. But instead of taking a point at pleasure, we may always assign for it such a position, that the plane of the couple shall be perpendicular to the axis of simple rotation; and then the whole motion is reduced to a rotatory motion about a determinate axis, and a motion of translation in the direction of this axis. Whence it results that the motion is identical with that of an external screw which turns in the corresponding internal screw. All the points in the body therefore describe on concentric cylinders small arcs of *helices* which have all the same *furrow* *. In the next instant it is a different screw, with another axis and a different *furrow:* and so on, whence we see the way in which are formed the simultaneous curves which all the points describe in space, and in which they move as in curvilinear canals, wherein we may suppose them to be inclosed.

Sometimes the *furrow* of this screw vanishes, and then the whole motion is reduced to a simple

* "*Pas*," the distance between two contiguous threads.

rotatory motion about the axis of the screw, which becomes what is called the *spontaneous axis* of rotation.

But in general the furrow of the screw does not vanish, and there is no spontaneous axis properly so called: that is, there is no straight line in the body, all the points in which remain immoveable for an instant. But there is always what we may call *a sliding spontaneous axis*, that is, a succession of points, forming a straight line, which have no motion other than a simple one of translation, in the direction of this line.

Such are the simplest notions and the clearest illustrations that we can form for ourselves of the motion of bodies. The mere motion of translation and the mere rotatory motion require no explanation to enable us to conceive them. Any motion whatever may always be reduced, and that in an infinite number of ways, to two such motions. And among this infinite number of reductions there is always one which furnishes the axis of rotation in the actual direction of the translation: so that, *the most general motion of which a body is capable is, as we before observed, that of a certain external screw which turns in the corresponding internal screw.*

After considering the actual motion of bodies in a point of view purely geometrical, I proceed to enquire into the forces capable of producing it, in order to determine conversely the motion due to any given forces; which is the natural object of the science of Dynamics.

Forces capable of producing a given Motion.

Whatever be the motion of a body there always exist forces capable of producing it. For at any instant during the motion of the body we may consider every molecule as if it were at rest, but acted on by a force capable of giving it the actual velocity which we suppose it to possess. Therefore an infinity of similar forces, applied individually to all the equal molecules of the body, are capable of producing in it the actual motion which we observe; and this too spontaneously, I mean to say even if these molecules have no mutual connexion, and consequently without causing any violent strain, which would tend to destroy their actual connection.

Such are the elementary forces capable of producing a given motion in a system of molecules, either free, or arbitrarily connected with each other.

But if these molecules constitute, as I here suppose, a system of invariable form, we may compound together all these elementary forces, and so replace them by a single force and a single couple, which will be equally capable of producing a given motion in the *solid body* under consideration. (10)

(10) This follows immediately from the theory of couples,* and includes d'Alembert's principle, which is, that "if any number of forces act upon a rigid system to produce motion, the elementary forces which indi-

* *Pritchard,* Prop. VII.

vidually propel every molecule of the system, and which are called the *effective* forces, are statically equivalent to these *impressed* forces." Since the system is rigid, we may reduce each set of forces to a single force applied at a given point, and a couple whose plane passes through this point. Suppose a set of forces, respectively equal and opposite to each of the *impressed* forces, to be applied at the same points; then these may be reduced to a single force, applied at the same point as before, and a couple whose plane also passes through this point. Now the force will clearly be equal to the resultant of the *impressed* forces, and in an opposite direction, and the couple will be equivalent to and in the same plane with the resultant couple of the impressed forces, and tend in an opposite direction. But such a set of forces would entirely prevent motion; therefore their resultant force and resultant couple must exactly balance the resultant force and resultant couple of the *effective* forces. Hence the impressed and effective forces are statically equivalent.

The impressed force at any one of the points of application may be more than adequate to produce the actual motion of the molecule on which it immediately acts, and hence we might imagine a part of it to be *lost*. On the other hand at some points the impressed forces may be inadequate to produce the actual motion, and at others there may be no impressed forces at all; whence arises an idea of forces being *gained* at these points. It follows immediately from what has been said above that "the forces lost and gained are statically equivalent," and this is the original form in which the principle was stated.

Let us enquire therefore what are respectively the force and the couple which together correspond to a given motion.

And first, if the body has only a *mere motion of translation* in space, so that all the equal molecules have velocities equal, parallel and in the same direction, it is manifest that all the elementary forces capable of producing these velocities are also equal, parallel, and in the same direction, and consequently reducible to a single force, parallel and in the same direction, and equal to their sum, applied at the centre of gravity of the body. Hence we see conversely, that *the effect of any force, applied arbitrarily at the centre of gravity of a body, is to transport all the particles thereof in its own direction, with a velocity measured by the magnitude of the force divided by the mass of the body*; which is, so to speak, self-evident.

In the second place, if the body has only a *mere rotatory motion* about any axis, it is evident that the velocities, and consequently the forces by which the individual molecules are impelled, are all proportional to the distances of the molecules from this axis, and in directions always perpendicular to these distances, and to the axis of which we speak. Now these elementary forces are always reducible to a single force and a couple. But if the axis passes through the centre of gravity of the body the force vanishes, and the whole is reduced to a couple, whose plane is in general inclined to this axis. If in addition the axis is one of the three rectangular ones which exist in all bodies, and which are called the three *principal axes*, we find that the couple is per-

pendicular to this axis, and that it is measured by the product of the angular velocity by the *moment of inertia* of the body about this principal axis. Whence we conclude conversely, that *the effect of a couple acting on a body in a plane perpendicular to one of its three principal axes is to make the body turn about this axis, with an angular velocity equal to the moment of the couple divided by the moment of inertia of the body about this axis.* (11)

(11) Let a rigid system turn about an axis OG (fig. 11.) with an angular velocity ω, and let any point P in it, whose co-ordinates are $ON = x$, $NP = y$, $OG = z$, be the position of a molecule whose mass is m, PT being a tangent at P to the circle which it describes about OG.

Then m is impelled by a moving force equal to the product of its mass and velocity $= m \,.\, \omega \,.\, OP$, applied at the point P in the direction TP; which we may resolve into two others at the same point, one

$$= m\omega \,.\, OP \,.\cos TPN = m\omega \,.\, OP \,.\sin OPN = m\omega x$$

in the direction NP or Gy, and another $= m\omega y$ in the direction NO, or $= - m\omega y$ in the direction Gx.

Now we may suppose the former force to act on the body at any point N in the line of its direction. Let two opposite forces, each equal and parallel to it be applied at O; then we shall have a force at O in the direction $Gy = \omega m x$, and a *couple* whose moment is $\omega m x^2$ in the plane NOP; and the former will give us in the same manner a force at G in the direction $Gy = \omega m x$, and a couple in the plane zy whose moment is $\omega m x z$. Proceeding in the same way with the other force, we obtain,

A force at G in the direction $Gx = -\omega m y$, a couple in the plane NOP which tends in the same direction with the former and whose moment is $= \omega m y^2$,

And a couple in the plane zx which tends in an opposite direction* to that in the plane zy, and whose moment $= -\omega m y z$ when estimated in the same direction. And the couples in the plane NOP may be removed into the plane xy. Hence the resolved parts of all the elementary forces may be summed up, and reduced to,

(i) Two forces applied at G; viz.

$\Sigma(\omega m x) = \omega \Sigma(m x)$ in the direction Gy,

and $-\omega \Sigma(m y)$ .... Gx.

(ii) Two couples; viz.

$\omega \Sigma(m x z)$ in the plane zy

$-\omega \Sigma(m y z)$ .... zx,

tending to make the body turn about some axis in the plane of xy.

(iii) Two couples; viz:

$\omega(\Sigma m x^2)$ and $\omega(\Sigma m y^2)$ in the plane xy.

If G be the centre of gravity

$$\Sigma m x = 0, \quad \Sigma m y = 0,$$

therefore the two forces vanish.

$$\text{If } \Sigma m x z = 0, \text{ and } \Sigma m y z = 0,$$

the resultant couple lies wholly in the plane of xy, and is equal to

$$\omega\left\{(\Sigma m x^2) + (\Sigma m y^2)\right\} = \omega\left\{\Sigma m(x^2 + y^2)\right\}$$
$$= \omega \,.\, \Sigma(m \,.\, OP^2);$$

* See Note (2).

which is called the *moment of inertia* of the body about the axis GO.

For an investigation of the number and directions of the axes for which, when they are severally taken as the axis of x, the quantities Σmxz and Σmyz separately vanish, and for the method of determining the moments of inertia of bodies, see Appendix.

Now this one theorem gives us immediately the effect of a couple acting on a body in any plane whatsoever. For whatever be the couple, we can always decompose it into three others, respectively perpendicular to the three principal axes of the body: and dividing each of them by the moment of inertia relative to its axis, we shall have the three angular velocities with which these three couples would tend to make the body turn about their respective axes. If then from the centre as origin, we take three lines in these directions to represent at once the axes and velocities of rotation, and complete the parallelopiped, we shall have in the diagonal the axis and velocity of the rotatory motion to which the proposed couple gives rise at the first instant*. The simplicity of this operation is evident.

Motion of a Body arising from any given Forces.

Whatever be these forces they may always be reduced to a single one, passing through the centre of gravity of the body, and a single couple.

* See Note (7).

Now the effect of the force is a mere motion of translation in the direction of this force, and the effect of the couple is to impress on the body a rotatory motion about a certain axis passing through the centre of gravity, whose direction is determined by what we have just observed.

But it will be seen from what immediately follows that we can express much more clearly the direction of the instantaneous axis, relative to the plane of the couple which gives birth to it.

The Central Ellipsoid of a Solid Body.

Since the three components of the proposed couple give three rotatory motions which are respectively proportional to these couples, and inversely as the moments of inertia about the three axes, we perceive from geometrical considerations that the axis of rotation is in the direction of the diameter conjugate to the plane of the couple, in an ellipsoid whose axes are reciprocally proportional to the square roots of the moments of inertia about these axes. (12)

(12) Let A, B, C, be the moments of inertia about the principal axes, and let the centre of gravity G (fig. 11.) be taken as origin, and these three rectangular axes as the axes of x, y, and z respectively. Take in them the lines

$$a = \frac{1}{\sqrt{n \,.\, A}}, \quad b = \frac{1}{\sqrt{n \,.\, B}}, \quad c = \frac{1}{\sqrt{n \,.\, C}},$$

and about these three lines as axes describe an ellipsoid.

Let α, β, γ, be the angles which the axis of a couple, whose plane passes through the origin, makes with the co-ordinate axes; and let M be its moment. Then the equation to the plane of the couple is*

$$x\cos\alpha + y\cos\beta + z\cos\gamma = 0\ldots\ldots\text{(i)},$$

and the resolved parts of it respectively perpendicular to the axes are

$$M\cos\alpha,\qquad M\cos\beta,\qquad M\cos\gamma;$$

and by the theorem laid down in page 29, if ω_a be the velocity generated by the couple $M\cos\alpha$ about the principal axis Ox,

$$\omega_a = \frac{M\cos\alpha}{A} = Mna^2\cos\alpha;$$

and similarly, $\omega_b = Mnb^2\cos\beta$, $\omega_c = Mnc^2\cos\gamma$;

and if $GP = p$ be the resultant axis of rotation, meeting the surface of the ellipsoid in the point P, whose co-ordinates are x', y', z', and inclined at angles α', β', γ' to the axes,

$$x' = p\cos\alpha' = p\frac{\omega_a}{\omega_p}\dagger,\qquad y' = p\frac{\omega_b}{\omega_p},\qquad z' = p\frac{\omega_c}{\omega_p},$$

and the equation to the tangent plane at P, which is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1,$$

becomes $x\cos\alpha + y\cos\beta + z\cos\gamma = \dfrac{\omega_p}{Mnp}\ldots\ldots\text{(ii.)}$,

wherefore the planes (i.) and (ii.) are parallel; that

* Hymers, Art. 3. Cor. 1.
† Note (7).

is, GP is the diameter conjugate to the plane of the couple*.

I suppose therefore an ellipsoid to be constructed about the centre of gravity of the body, having its three principal axes in the directions of the principal axes of the body, the squares of their lengths being reciprocally proportional to the moments of inertia of the body about them: and I may here remark that this ellipsoid will possess the remarkable property, that the moment of inertia of the body about any one of its diameters will be inversely as the square of the length of this diameter. (13)

(13) The equation to a plane perpendicular to GP (v. last note) is

$$\delta = x \cos \alpha' + y \cos \beta' + z \cos \gamma';$$

and if Q be any point in this plane, the distance of Q from $GP = \sqrt{GQ^2 - \delta^2}$; therefore moment ($P$) of the body round GP, which $= \Sigma$ mass of a particle $\times$ (distance from GP)^2,

$$= \Sigma m (x^2 + y^2 + z^2 - \delta^2).$$

And since the co-ordinate axes are by hypothesis principal axes,

$$\Sigma myz = 0, \qquad \Sigma mxz = 0, \qquad \Sigma mxy = 0;$$

$$\therefore\ P = (\Sigma mx^2) \sin^2 \alpha' + (\Sigma my^2) \sin^2 \beta' + (\Sigma m z^2) \sin^2 \gamma',$$

$$\text{or since } \sin^2 \alpha' = 1 - \cos^2 \alpha' = \cos^2 \beta' + \cos^2 \gamma',$$

$$P = \cos^2 \alpha' . \Sigma m (y^2 + z^2) + \cos^2 \beta' . \Sigma m (x^2 + z^2)$$
$$+ \cos^2 \gamma' \Sigma m (x^2 + y^2)$$

* Hymers, Art. 24. Cor. 1.

$$= A\cos^2\alpha' + B\cos^2\beta' + C\cos^2\gamma'$$

$$= \frac{1}{na^2}\cos^2\alpha' + \frac{1}{nb^2}\cos^2\beta' + \frac{1}{nc^2}\cos^2\gamma'$$

$$= \frac{1}{na^2}\cdot\frac{x'^2}{p^2} + \frac{1}{nb^2}\cdot\frac{y'^2}{p^2} + \frac{1}{nc^2}\cdot\frac{z'^2}{p^2}$$

$$= \frac{1}{np^2}, \quad \text{since } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1.$$

Now whatever be the form and constitution of a body, it has always a centre of gravity and three principal axes, as distinctly determinate as those of any homogeneous regular solid*. We may therefore always conceive such an ellipsoid as the above to be constructed in it, which has the triple advantage, of placing before our eyes the centre of gravity and principal axes, of giving us all the moments of inertia which we may have to take into consideration (since the moment about any diameter is expressed in a simple function of its length), and of offering us an easy method of determining the position of the axis of instantaneous rotation, relative to the plane of the impressed couple. Now it is this ellipsoid, the consideration of which will throw great light on the theory of rotatory motion, that I shall for the future denominate the *central Ellipsoid.*

* See Appendix.

The Motion of a Body arising from any given Couple clearly expressed.

Suppose that a body at rest is struck by a couple in any plane drawn through the centre of gravity, which may therefore be considered a diametral plane of the central ellipsoid: we have just seen that the instantaneous axis of rotation to which the couple gives rise is the diameter *conjugate* to the plane of the couple: and the angular velocity will evidently be measured by the moment of the couple, resolved perpendicularly to this diameter, and divided by the moment of inertia of the body about the same diameter. (14)

(14) This may be proved strictly from the preceding notes. For if θ be the angle between the instantaneous axis and the axis of the couple,

$$\cos\theta = \cos\alpha \, . \cos\alpha' + \cos\beta \, . \cos\beta' + \cos\gamma \cos\gamma'$$

$$= \frac{\omega_a}{Mna^2} . \frac{x'}{p} + \frac{\omega_b}{Mnb^2} . \frac{y'}{p} + \frac{\omega_c}{Mnc^2} . \frac{z'}{p}$$

$$= \frac{\omega_p}{Mnp^2} \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} \right),$$

$$\therefore \ \omega_p = Mnp^2 \cos\theta = \frac{M \cos\theta}{P}.$$

It is manifest also from note (10) that the *impressed* couple M is equal to, and in the same plane with, the resultant couple of the *effective* forces; that is, the resultant of the two sets of couples (ii) and (iii) in note (11).

And resolving M parallel and perpendicular to the plane of the couples (iii), we shall have

$$M \cos \theta = \text{resultant of (iii)} = P \cdot \omega_p,$$

the same equation as before.

Likewise, $M \sin \theta =$ resultant of the couples (ii).

Hence the axis of an impressed couple can never be the corresponding axis of instantaneous rotation, unless it coincides with a principal axis of the body. For if $\theta = 0$, no part of M can be employed to produce the couples (ii), which form a necessary part of the *effective* forces, requisite about any axis not a principal one to give to each molecule its proper motion. The theorem (page 29) expressed by $M = P \cdot \omega_p$ only holds therefore for a principal axis.

Since a couple may always be removed into any plane parallel to its own, without causing any change in its effect on the body, we may always suppose the plane of the impressed couple instead of being drawn through the centre, to touch the surface of the ellipsoid: and then we may say, that

If a body is struck by a couple situated in any plane which touches the central ellipsoid of the body, the instantaneous pole of the rotatory motion to which the couple gives birth is the point of contact. And conversely that

If a body turns on any diameter of its central ellipsoid, the couple actually impressed on it is in the tangent plane at the pole.

Which appears to be one of the simplest theorems in Dynamical Science, on the difficult and obscure theory of rotatory motion.

We proceed to enquire how the rotatory motion changes from one instant to another, and to trace throughout its whole course the motion of the body.

SECTION II.

SOLUTION OF THE PROBLEM OF THE ROTATION OF FREE BODIES.

IT is evident that the axis of rotation which we denominate *the instantaneous axis* remains immoveable only for a single instant. For from the rotatory motion itself there arise, for each of the equal molecules of the body, centrifugal forces respectively proportional to, and in the direction of, the radii of the circles which these molecules tend to describe. Now the axis of which we speak being, by hypothesis, not a principal axis, these centrifugal forces will not balance each other. When removed parallel to themselves to the centre of gravity, they give it is true a resultant which equals nothing, but the resultant couple does not vanish. There arises therefore from the rotatory motion itself an *accelerating* couple, the action of which impresses on the body at each instant an infinitely small rotatory motion, which is compounded with the actual rotatory motion of the body, and causes both the magnitude and the axis of it to vary. (15)

(15) The *whole* effect of the couple being, as we saw in the last note, to produce a rotatory motion about GP, with an angular velocity $\omega_p = \dfrac{M\cos\theta}{P}$, we may consider the body in motion as acted on by no external force. And in such case the tension of the rod (QR),

connecting the molecule m at Q with the axis GP, which is for the instant immoveable,

$$= m \cdot \frac{(\text{velocity of } Q)^2}{QR}$$

$$= m \cdot QR \cdot (\omega_p)^2$$

applied at R in the direction RQ; which is equivalent to

a force $= (\omega_p)^2 \cdot mQR$ at G, perpendicular to GP,

and a couple whose moment is $(\omega_p)^2 \cdot mQR \cdot GR$, and whose plane passes through GR.

And it is clear from the reasoning in note (11) that the resultant of all these centrifugal forces at G will vanish, by the property of the centre of gravity.

And the resultant of all the couples will be a couple, in a plane passing through the axis, equal to

$$\omega_p \times (\text{resultant of all the couples } \omega_p \cdot mQR \cdot GR)$$

$$= \omega_p \times \{\text{resultant of the couples (ii) for the axis } GP\}$$

which does not vanish; since the axis is not a principal one.

In order to investigate the motion of the body, it is necessary therefore to commence by finding this accelerating couple which arises from the centrifugal forces. Now it is very easy to *see directly*, or to conclude from the principle of the *conservation of couples**, that if we take two lines to represent respectively, the axis and magnitude of the impressed couple, and the axis and magnitude of the instantaneous rotatory motion, the accelerating couple due to the centrifugal forces is always represented in magnitude and position

* See Appendix.

by the surface of the parallelogram constructed on these two lines: a simple and remarkable theorem, which includes the whole theory of rotatory motion, and which when interpreted analytically gives immediately those three elegant equations which we owe to Euler, but which are ordinarily obtained by long and circuitous operations only. (16)

(16) It is clear that since each centrifugal force is proportional, and in a direction perpendicular, to the corresponding effective force which must have originally acted on the particle m to produce its motion, the resultant couple will be in a plane perpendicular to that of the resultant couple of the effective forces, and therefore, during the first instant, perpendicular to that of the impressed couple M.

The plane of the accelerating couple will therefore, during the first instant, pass through the axis of M as well as through the axis of rotation.

And if (l) be a linear unit, we may take GP (fig. 12.) $= l \,.\, \omega_p$,

$$\text{and } GM \text{ in the axis of the couple} = \frac{M}{l}.$$

Then the area of the parallelogram Gm, which is manifestly in the plane of the accelerating couple,

$$= GP \,.\, GM \,.\, \sin PGM$$
$$= \omega_p \,.\, M \sin\theta\,;$$

which from notes (14) and (15) we know to be the magnitude of the couple, *during the first instant.*

And since the body is acted on by no external force, whatever be the position into which it is brought by the action of the centrifugal forces, we may assume that

stantaneous rotation; and the height of it above GQ is manifestly the same as that of P.

$$\text{Also velocity round } GR = \omega_p \frac{GR}{GP},$$

we see moreover from the reasoning in note (16) and from the theorem in page 37, that the body will come into such a position that the tangent plane at the pole R shall be parallel to the plane of the couple originally impressed: which in the original position of the body we may suppose to be a tangent at P.

But the centre is fixed in absolute space, and the plane of the couple always parallel to itself; therefore *this plane, which is always a tangent at the instantaneous pole of rotation, is an invariable plane, fixed in absolute space.*

Therefore the motion of the body, or what is the same thing the motion of the central ellipsoid, is such, that this ellipsoid remains in contact with a plane fixed in absolute space: that is, it turns at every instant about the radius vector at the point of contact, with an angular velocity proportional to the length of this radius.

This ellipsoid therefore rolls without sliding on the fixed plane abovementioned; for since all its motion consists in turning for an instant on the line drawn from the centre to the point of contact, the ellipsoid brings at the end of this instant a new point into contact with the plane; and this new point, which becomes the pole of rotation for the instant following, remains in its turn immoveable for this instant, and so on to infinity; whence it is manifest that none of the

points at which the ellipsoid comes into contact with the fixed plane can ever slide on this plane.

We have therefore a clear idea of the complicated motion of a body of any shape whatever, turning freely either on its centre of gravity or on any other fixed point, due to the action of a couple whose impulsion it originally received in any given plane. For about the center of gravity, or if the body is not free, the fixed centre of rotation, and the principal axes of the body at that point, we may describe the *Central Ellipsoid,* to which we may confine our attention and neglect entirely the shape of the body. We may then suppose this ellipsoid whose centre is fixed, to roll without sliding on a fixed plane touching it, which gives us an exact representation of the geometrical motion of the body: and since the angular velocity with which it turns about the radius at the point of contact is proportional to the length of this radius, we obtain also the dynamical motion of the body; that is, we have before us the complete succession of the positions which the body occupies, and the time which it takes to pass from any one to any other; which completely determines the motion.

SECTION III.

DEVELOPEMENT OF THE SOLUTION.

The two Curves described by the Instantaneous Pole of Rotation.

THE above illustration of the rotatory motion of a body leads us at once, and as it were by the hand, to the calculations necessary to measure all the different affections of this motion.

And first this succession of points, at which the central ellipsoid comes into contact with the fixed plane of the impressed couple, traces on the surface of the ellipsoid the path of the instantaneous pole in the interior of the body, and the corresponding succession of points on the fixed plane traces its path in absolute space. We can therefore determine immediately these two curved lines, and consider them as the bases of two conical surfaces having the same vertex, one of which, moving with the body, would by rolling on the other, which is fixed in absolute space, cause in the body the precise motion with which it is endued.

To find the first curve we have only to determine the succession of points in which the ellipsoid is touched by a plane which is always at the same distance from its centre; or what is the same thing, which touches a concentric sphere whose radius is equal to the given distance.

While this plane traces on the ellipsoid the path of the instantaneous pole, we may remark that it traces on the sphere the path that the pole of the couple, which is fixed in space, would appear to describe in the interior of the moveable body; a curve of the same nature which we shall have also occasion to consider.

But to speak of the first only: we see that it is *a re-entering curve of double curvature,* having like the ellipse *four principal vertices,* at which it is divided into four equal and symmetrical parts; a species of elliptical *wheel,* whose *axle* is always either the *greatest* or *least radius* of the central ellipsoid, according as the radius of the sphere is given greater or less than the mean radius of the ellipsoid. This curve of double curvature is projected in *a complete ellipse* on the plane perpendicular to the axis which forms its *axle,* in *an elliptic arc* on the other plane, and always in *an hyperbolic arc* on the plane perpendicular to *the mean radius.* (19)

(19) Let Pg (fig. 14.) be a perpendicular section of the tangent plane touching the ellipsoid in P and a concentric sphere whose radius $Gg = r$ in g.

Let x, y, z, be the co-ordinates of P,

x', y', z', ... of g,

x'', y'', z'', ... of any point in the tangent plane.

Then we have two equations to this plane; viz:

$$\frac{xx''}{a^2} + \frac{yy''}{b^2} + \frac{zz''}{c^2} = 1,$$

$$\text{and } \frac{x'x''}{r^2} + \frac{y'y''}{r^2} + \frac{z'z''}{r^2} = 1,$$

which must coincide; therefore $\dfrac{x'}{r^2} = \dfrac{x}{a^2}$,

$$\text{and } \frac{x'^2}{r^4} = \frac{x^2}{a^4},$$

$$\frac{y'^2}{r^4} = \frac{y^2}{b^4},$$

$$\frac{z'^2}{r^4} = \frac{z^2}{c^4};$$

$$\text{Therefore } \frac{1}{r^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4};$$

$$\text{Also } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

which are the equations to the curve of double curvature. Let a be the greatest and c the least semi-axis. Then if $r > b$, the curve can never meet the plane perpendicular to the major axis;

$$\text{for if } x = 0, \quad z^2 = c^2.\frac{\dfrac{1}{r^2} - \dfrac{1}{b^2}}{\dfrac{1}{c^2} - \dfrac{1}{b^2}},$$

a negative quantity. The curve therefore lies in this

case wholly about the major axis. And *vice versâ* if $r < b$. Also the equation to the projection on the principal plane perpendicular to c is

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + c^2\left(\frac{1}{r^2} - \frac{x^2}{a^4} - \frac{y^2}{b^4}\right),$$

$$\text{or } \frac{x^2}{a^2} \cdot \frac{1 - \frac{c^2}{a^2}}{1 - \frac{c^2}{r^2}} + \frac{y^2}{b^2} \cdot \frac{1 - \frac{c^2}{b^2}}{1 - \frac{c^2}{r^2}} = 1,$$

which is evidently the equation to an ellipse whose semi-axes are

$$a\sqrt{\frac{1 - \frac{c^2}{r^2}}{1 - \frac{c^2}{a^2}}}, \quad \text{and } b\sqrt{\frac{1 - \frac{c^2}{r^2}}{1 - \frac{c^2}{b^2}}},$$

the first of which is always $< a$; and if $r < b$, the second is $< b$, or the projection is a complete ellipse, concentric to the principal section of the ellipsoid; but if $r > b$ the minor semi-axis is $> b$, and the projection is only part of an ellipse.

The equation to the projection perpendicular to the mean semi-axis is

$$\frac{x^2}{a^2} \cdot \frac{1 - \frac{b^2}{a^2}}{1 - \frac{b^2}{r^2}} + \frac{z^2}{c^2} \cdot \frac{1 - \frac{b^2}{c^2}}{1 - \frac{b^2}{r^2}} = 1,$$

which is manifestly the equation to an hyperbola, the coefficients of x^2 and z^2 having necessarily different signs.

The four vertices of this curve are the points where the radius vector, and consequently the velocity of rotation, attain their *maximum* and *minimum* values; and we may remark that the *maximum* always occurs when the instantaneous pole passes through the two vertices which lie in the *mean* principal plane of the ellipsoid, and the *minimum* when it passes through the other two vertices. (20)

(20) Since $GP = \sqrt{x^2 + y^2 + z^2}$,

$$0 = d_x GP = GP \, . \, d_x GP$$

$$= x + y d_x y + z d_x z,$$

also $$0 = \frac{x}{a^2} + \frac{y}{b^2} d_x y + \frac{z}{b^2} d_x z,$$

$$0 = \frac{x}{a^4} + \frac{y}{b^4} d_x y + \frac{z}{b^4} d_x z,$$

whence we derive separately $y = 0$, $x = 0$, which give possible values for GP when $r > b$, the former satisfying the conditions for a maximum, and the latter for a minimum; the value of the maximum radius (P) being

$$\sqrt{x^2 + z^2} = \frac{a^2 \sqrt{r^2 - c^2} + cr\sqrt{a^2 - r^2}}{r\sqrt{a^2 - c^2}},$$

and of the minimum (ρ),

$$\sqrt{x^2 + y^2} = \frac{a^2 \sqrt{r^2 - b^2} + br\sqrt{a^2 - r^2}}{r\sqrt{a^2 - b^2}}.$$

The second curve, being traced by that which rolls about the centre on the fixed plane of the couple, is therefore a plane curve which encircles

the projection of the centre, forming equal and regular undulations corresponding to the equal and symmetrical arcs of the rolling orbit which produces it: it is a species of circular curve whose radius varies periodically, and which winds for ever between two concentric circles whose circumferences it touches alternately. (21)

(21) The projection (gp) of the radius vector GP on the fixed plane of the couple will evidently arrive at its maximum and minimum values cotemporaneously with GP; for when GP is greatest its inclination to this plane is least.

With centre g, (fig. 15.) the projection of G, and radii ρ and P, describe two circles; then the curve being perpendicular to the radius vector gp at the points where it attains its maximum and minimum values, will manifestly touch these circles at those points; that is, will touch them alternately, since the curve passes alternately the mean and major principal planes.

The consequences deduced in this and the last note for $r > b$ are easily adapted to the case when $r < b$.

If the angle at the centre which corresponds to two consecutive vertices of these equidistant undulations is commensurable with four right angles, the curve re-enters itself after a certain number of revolutions; and the instantaneous pole which describes it returns at once to the same position, both in the body and in space. But in the contrary case the curve never re-enters itself, and the pole, which always returns periodically to the same place in the body, can never

return at the same time to the same point in space.

Such are the two curves described by the instantaneous pole, the one in the interior of the body, and the other in absolute space. And although these curves are of such different forms, yet since it is one and the same point which describes them both, the equations to them, between the radius vector and the arc, exactly coincide. (22)

(22) To find this equation to the two curves, we have

$$p^2 = x^2 + y^2 + z^2,$$

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

$$\frac{1}{r^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}.$$

Substituting in the second and third the values of z^2 derived from the first, multiplying the former by $\frac{1}{b^2} + \frac{1}{c^2}$, and subtracting, we have

$$x^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right)\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + \frac{p^2}{b^2c^2} = \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{r^2};$$

$$\therefore\ x^2 = a^4 . \frac{\frac{b^2c^2}{r^2} - \left(\frac{1}{b^2} + \frac{1}{c^2}\right) + p^2}{(a^2 - b^2)(a^2 - c^2)},$$

or $x = a'.\sqrt{\alpha^2 + p^2}$, suppose;

$$\therefore\ d_p x = \frac{a'p}{\sqrt{a^2 + p^2}}.$$

$$\text{Similarly, } d_p y = \frac{b'p}{\sqrt{\beta^2 + p^2}}, \quad d_p z = \frac{c'p}{\sqrt{\gamma^2 + p^2}};$$

$$\therefore\ d_p s, \text{ which} = \sqrt{(d_p x)^2 + (d_p y)^2 + (d_p z)^2}$$

$$= p\sqrt{\frac{a'^2}{a^2 + p^2} + \frac{b'^2}{\beta^2 + p^2} + \frac{c'^2}{\gamma^2 + p^2}}.$$

The *rolling cone* of which the first curve forms the base is simply *a right cone of the second degree;* but the *fixed cone* on which it rolls is a *transcendent cone,* whose surface undulates for ever about the fixed axis of the couple: it is a species of right circular cone, whose surface however is *fluted* according to the regular undulations of the curve which forms its base.

Proposed Names for the two Curves.

We know that a heavy body projected any how in space turns on its centre of gravity, exactly as if it were free from the action of gravity. The two remarkable curves therefore above described are presented constantly to our notice in the motion of projectiles, and merit names as much as the path of the centre of gravity which is called a parabola.

I propose therefore to give them the names of relative and absolute *Poloids;* or rather, in order to distinguish them by their respective forms,

to call the first simply the *poloid*, and the second the *serpoloid*.

It is evident that the forms of the curves will depend entirely on four given quantities, viz. the three semi-axes of the central ellipsoid which are always given by the nature of the body, and the height of the centre above the tangent plane of the couple, which is given by the direction of the impressed couple.

Particular Case in which the Poloid becomes an Ellipse, and the Serpoloid a Spiral.

In the particular case in which the height above the plane of the couple is exactly equal to the *mean radius* of the ellipsoid, the poloid becomes an *ellipse*, whose plane passes through the mean radius, and the serpoloid a spiral, which when examined throughout its whole extent appears to be a sort of *double spiral*. I mean that it throws out on opposite sides of a vertex two equal branches whose generating points revolve in opposite directions about a fixed centre, which they continually approach, but which is a species of *asymptotic point* that they never attain. In this case therefore the instantaneous pole is always a new point both in the body and in absolute space, although the length of the spiral described is finite, and moreover equal to the semi-circumference of the rolling ellipse which produces it. (23)

(23) When $r = b$, we have

$$\frac{x^2}{a^2} \cdot \frac{b^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \cdot \frac{b^2}{c^2} = 1,$$

$$\text{and } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\text{whence } x = \frac{a^2}{c^2} \cdot \sqrt{\frac{b^2 - c^2}{a^2 - b^2}} \cdot z,$$

and the curve lies wholly in a plane passing through the axis of y. It is therefore an ellipse whose axes are

$$\sqrt{\frac{a^2b^2 + b^2c^2 - a^2c^2}{a^2 - b^2}} = a' \text{ and } b.$$

Now the latter is evidently the value of Gg (fig. 14.) in this case. And as the value of GP diminishes from the former, the angle PGg, which is the angle (θ) between the instantaneous axis and the fixed axis of the couple, perpetually diminishes, and the accelerating couple arising from the centrifugal forces ($\omega_p . M \sin \theta$) becomes less than any assignable quantity; that is, no finite couple M can make the pole coincide with g (fig. 16.) though it continually approaches it in moving either way from K, which lies in the circumference of the circle described with radius $GK = a'$ on the plane of the couple.

Particular Cases in which they are reduced to a Point.

If the distance of the centre from the tangent plane is given equal to one of the *extreme radii* of the ellipsoid, then since this can only happen at a single point in the surface, the poloid and ser-

poloid are both reduced to a single point, and the instantaneous pole remains immoveable both in the body and in space during the whole motion; and the same thing will happen in a single variety of the particular case before considered, namely, when the plane of the couple touches the central ellipsoid at its mean pole. (24)

(24) In all these cases the axis of the impressed couple coincides with a principal axis, and therefore the theorem of page 29 is applicable, and the axis of the couple being the instantaneous axis, $\theta = 0$ and the centrifugal couple vanishes, that is, the elementary centrifugal couples mutually destroy each other.

Particular Case, arising from the Constitution of the Body, in which the two Curves become Circles.

Lastly, if the body is one of those which have two of their principal moments of inertia equal to one another, in which case the ellipsoid becomes a solid of revolution, the poloid becomes a circle about the axis of this spheroid, and the serpoloid another circle about the fixed axis of the couple. In all bodies of this kind the movement is that of a right cone whose base is a circle rolling uniformly on a fixed cone of the same kind. It is one of the simplest cases of rotatory motion, but we must remark that if, as is usually the case, the circumferences of these circles are incommensurable, the instantaneous pole can never return at the same time to the same point in the body and in absolute space. (25)

(25) If $a = b$, the equations to the poloid become

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1,$$

$$\text{and } \frac{x^2 + y^2}{a^4} + \frac{z^2}{c^4} = \frac{1}{r^2},$$

whence we have

$$\frac{z^2}{c^2} \cdot \left(\frac{1}{c^2} - \frac{1}{a^2}\right) = \frac{1}{r^2} - \frac{1}{a^2},$$

$$\text{and } \frac{x^2 + y^2}{a^2} \cdot \left(\frac{1}{c^2} - \frac{1}{a^2}\right) = \frac{1}{c^2} - \frac{1}{r^2},$$

the equations to a circle in a plane perpendicular to the axis of z.

Hence GP, and therefore the angular velocity, is constant.

And the serpoloid being traced out by P (fig. 14.), which always remains at the same distance $\sqrt{GP^2 - r^2}$ from g, will also be a circle.

The uniformity of the motions of the pole in the body and in space will be shewn hereafter.

We need not examine the simplest case of all, in which the central ellipsoid is a perfect sphere: for in whatever manner the couple is applied, the axis of rotation and the axis of the couple coincide, and the instantaneous pole remains immoveable in the body and in absolute space.

Velocities, with which the Pole describes the two Curves, with which it approaches and recedes from the Centre, with which it revolves about the fixed Axis of the Couple, &c.

After having examined the nature of the two curves described by the pole, which have the same differential equation between the radius vector drawn from the centre of the ellipsoid and the arc, we may consider the velocity with which the pole describes them both; that with which it recedes from or approaches the centre, which is the increment or *fluxion* of the radius vector and consequently of the angular velocity; and the angular motion of the pole about the fixed axis of the impressed couple. It is easy to determine the remarkable points where these different velocities have their *maximum* and *minimum* values, which occur at the alternate vertices of the waves of the serpoloid. But we may make a curious remark on the velocity of the pole along this curve; viz. that it may have, in certain cases, a *minimum* value at the *superior* vertex of the wave, and likewise a *minimum* at the *inferior* vertex, and consequently a *maximum* at a third point intermediate to them; which will not hold for the other velocities, whose maximum and minimum values occur only at the vertices of the curve described by the pole. (26)

(26) The velocity in the poloid $(d_t s) = PR$ (fig. 13.)

$$= \frac{p}{\omega_p} \text{(velocity round } GQ\text{)};$$

and since the plane of the centrifugal couple passes through the axis of the impressed couple, the axis GN of the former lies in the plane of the latter:

$$\text{and } \omega_q = \frac{(\text{moment of centrifugal couple}) \cos NGQ}{\text{moment of inertia round } GQ}$$

$$= \frac{\omega_p \,.\, M \sin\theta \,.\, \cos NGQ}{Q};$$

$$\therefore\; d_t s = Mnpq^2 \sin\theta \cos NGQ.$$

But by Napier's rules,

$$\cos PGQ = \cos PGU \,.\, \cos QGU,$$

$$\text{and } \cos PGU = \sin\theta = \frac{\sqrt{p^2 - r^2}}{p},$$

$$\cos PGQ = -\, d_s p\,^*,$$

$$\text{whence } \cos QGN = \left\{1 - \frac{(pd_s p)^2}{p^2 - r^2}\right\}^{\frac{1}{2}},$$

$$\text{and } (pq)^2 \sin^2 PGQ + (qu)^2 \sin^2 UGQ + (pu)^2 \sin^2 PGU$$

$$= (ac)^2 + (ab)^2 + (bc)^2,$$

$$\text{also } p^2 + q^2 + u^2 = a^2 + b^2 + c^2,$$

whence we may obtain q in terms of p and $d_s p$; and substituting, we have $d_t s$ in terms of the same quantities. But from note (22) we have $d_s p = \dfrac{1}{pF(p)}$, and we can therefore determine $d_t s$ in terms of p alone.

And since $\omega_p = Mnp^2 \cos\theta = Mnpr$,

$$d_t \omega_p = Mnr \,.\, d_t p = Mnr\, d_t s \,.\, d_s p,$$

* Miller's *Differential Calculus*, Art. 94.

which is reducible to an elliptic function of the kind treated by Legendre, Chap. XXXIII.

If ρ be the radius vector Gg of the serpoloid,

$$p = \sqrt{r^2 + \rho^2},$$

and $$d_\rho s = d_p s \,.\, d_\rho p = \frac{\rho}{\sqrt{r^2 + \rho^2}} \,.\, \sqrt{r^2 + \rho^2} \,.\, F(r^2 + \rho^2)$$

$$= \rho F(r^2 + \rho^2),$$

$$d_t s = \rho F(r^2 + \rho^2)\, d_t \rho,$$

and at the vertices $d_t \rho = 0$, $d_\rho s = \infty$.

We may afterwards simplify the equation to the serpoloid by referring it to the radius vector drawn from its own centre in the plane in which it lies, and the angle described by this radius about the centre. And any such expression may be obtained without difficulty. (27)

(27) If θ be the angular distance of the radius vector from a given position,

$$(d_\rho \theta)^2 = \frac{(d_\rho s)^2 - 1}{\rho^2} = \{F(r^2 + \rho^2)\}^2 - \frac{1}{\rho^2},$$

which is reducible as before to an elliptic function.

Determination of the Position of the Body at the end of a given Time.

Lastly, in order to find the formulæ which will enable us to calculate the position of the body at the end of a given time, we must begin by determining the velocity of rotation in terms of

the time, which will be done by a single integration, and will give the place of the instantaneous pole on the surface of the central ellipsoid. Afterwards we must integrate the above equation to the serpoloid, which will give the place of the pole on the fixed plane of the couple. And by these two quadratures, which naturally belong to the class of *elliptic transcendents*, we may say that the proposed problem is entirely solved; I mean that we are enabled by means of them to determine the actual position in space in which a body is found at the end of a given time. For we have only to suppose the ellipsoid placed in contact with the fixed plane, so that they touch at the points which we have just determined in the ellipsoid and plane respectively; when the central ellipsoid, and consequently the body, will have the exact position in space at which it arrives by its natural motion at the end of the given time.

We may vary these determinations in different ways, by taking other unknown quantities relative to the position of the body; but whatever be the co-ordinates which we employ, the expression of these quantities in terms of the time will always require two integrations, which necessarily belong to the class of elliptic transcendents.

In the particular case in which the height of the centre above the fixed plane is equal to the mean radius of the ellipsoid, when the poloid becomes a simple ellipse and the serpoloid a spiral, the difficulty is diminished, and the integrals become *logarithmic* or *exponential*. (28)

(28) In this case $d_p s$ is reducible at once to an elliptic function; and the value of q^2 deduced from note (26) is greatly simplified; and thus the values of $d_t \omega_p$ and $d_\theta \rho$ are reduced as here stated.

Lastly for bodies whose central ellipsoid is one of revolution, when the motion becomes wholly uniform and circular, no integration is required to determine the position of the body at the end of a given time. (29)

(29) In this case all the axes in the equatoreal plane AGB (fig. 17.) of the spheroid are principal axes. Let AGC be the plane passing through the axis of the spheroid and the axis of the couple, GB the intersection of the equatoreal plane with the plane of the couple, which will be manifestly perpendicular to AGC. Then the resolved part of M perpendicular to the principal axis $GB = 0$, and therefore the axis of instantaneous rotation lies in the plane AGC.

Therefore AGC is the plane of the centrifugal couple, and GB, which is the diameter conjugate to it, the axis about which this couple tends to make the body turn.

And from note (25) GP $(=p)$ is constant, as indeed appears also from the consideration that GB and there-the small arc PR is perpendicular to GP; substituting therefore in the equation obtained in note (26) and observing that $NGQ = 0$, we have

$$d_t s = Mnpa^2 \frac{\sqrt{p^2 - r^2}}{p}, \text{ a constant quantity.}$$

For the application to the theory of Precession, see Appendix.

Properties of the three principal Axes of the Body, relative to the Stability of the Rotatory Motion.

When the plane of the impressed couple is so situated that the instantaneous pole falls on one of the principal poles of the central ellipsoid, it always remains there; so that the instantaneous axis is the immoveable straight line in which the axes of the body and the couple coincide throughout the whole motion. Each of the three principal axes is therefore a permanent axis of rotation. But there exists between them, as is known, a remarkable difference with respect to the stability of the rotatory motion about each.

If the instantaneous pole falls on the *greater* or *lesser* pole of the ellipsoid, and happens by the impulsion of any small disturbing couple to be drawn aside to a little distance therefrom, it will not recede farther, but will describe its poloid about this particular pole of the ellipsoid. But this is not the case when the instantaneous pole falls on the *mean* pole of the ellipsoid; for on any the slightest displacement it will recede farther and farther, and proceed to describe its poloid about the greater or lesser pole, according as the accidental disturbance causes the distance of the tangent plane of the couple from the centre of the ellipsoid to increase or diminish. And if the disturbance is such that this distance is not altered, which happens in the directions of two particular ellipses which cross each other at the mean pole, the in-

stantaneous pole will proceed to describe the particular ellipse along which it has been started, or rather the half of this ellipse, till it arrives at the opposite mean pole; which is the greatest disturbance that the body can suffer: whilst, if it had been started along the other half of the ellipse it would have returned immediately to the same mean pole; which is the least possible disturbance. There is therefore a single case in which the instantaneous axis being drawn aside from the mean axis, with which it coincided at first, not only does not recede farther from it, but even returns towards it immediately, until its distance is less than any assignable quantity. But in all other cases it proceeds to describe an elliptical cone about the major or minor axis, or to trace the plane of one or other of the ellipses which I have mentioned: and we may say that the rotatory motion about the mean axis has no stability. (30)

(30) The rotatory motions about GP, GQ (fig. 13.) are supposed to be in the same direction; and in that case P moves in the *direction* MPU. Suppose the direction of the motion which the impressed couple tends to produce about its own axis to be reversed; then the motion about the instantaneous axis will be reversed, the motion which the centrifugal couple tends to produce about GQ being unchanged. The motions about GP, GQ will therefore be in opposite directions, and in order to find R the new pole we must take q on the other side of G. The pole will therefore move in the direction UPM.

Hence the direction of the motion of the pole in the poloid depends on the direction of the motion which the

impressed couple tends to produce about its axis. If therefore, for a given direction of this motion, the instantaneous pole, on being started in one direction from the mean pole along the above-mentioned ellipse, tends to describe the whole semi-ellipse, it will on being started along the other half of this ellipse still tend to move in the same direction as before, i. e. will return towards the mean pole; which from note (24) it will approach nearer than by any assignable quantity.

The only axes of stable rotatory motion therefore are those about which the moments of inertia of the body are respectively the greatest and least; but we must not conclude that it is equally stable about these two axes: for if one of them differs little from the mean axis, the stability of the motion about it will not be greater than of that about the mean axis, as we shall presently see.

Measure of the Stability about each of the two extreme Axes.

In order to form a distinct idea of this stability, and of that which forms in some degree the measure of it for each of these two axes, imagine the surface of the ellipsoid to be cut into four elliptical sections by the two ellipses above-mentioned, whose planes intersect in the mean axis. The mean pole is therefore at the intersection of these two ellipses; the greater pole is in the centre of one of a pair of sections, and the lesser pole in the centre of the supplementary section.

Now, in the first place, if the instantaneous pole of rotation falls on the mean pole of the ellipsoid, it is clear that if disturbed ever so little it will be thrown into one or other of these two sections, and describe its poloid about one or other of the principal poles of the ellipsoid: or else if the disturbance is in the direction of one of the two ellipses it will either describe the half of this ellipse, or return immediately to its former position; this being the only case of stability about the mean axis.

Again, if the instantaneous pole falls on the *greater* pole of the ellipsoid it may be removed at pleasure to any point in the surrounding section without ceasing to describe its poloid about the same pole: and if it is in this that we make the stability about the major axis to consist, we may say that the magnitude of the section is in some degree the measure of it. Similarly we perceive that the supplementary section is the measure of the stability of the rotatory motion about the minor axis. Now if one of these two axes differ little from the mean axis, the corresponding section is very small, and the supplementary section very great. The axis which differs little from the mean axis affords therefore very little stability, and the other axis very much. It is not therefore correct to say, as people usually do, that if the instantaneous axis is drawn a little aside from the principal axis which corresponds to the greatest or least moment of inertia of the body, it recedes very little from it, and makes only small oscillations during

the whole period of the motion: for if the moment of inertia relative to this axis differs little from the mean moment, the instantaneous pole may be thrown by a small disturbance out of the little section in which it lies into the neighbouring section, and proceed to describe therein its poloid about the other axis; or else, if it is only removed to another point in this little section, it may describe therein a narrow and elongated poloid, and consequently make considerable oscillations about the principal pole from which it has been drawn aside. (31)

(31) If $r > b$, the semi-axes of the ellipse which is the projection of the poloid on the plane perpendicular to the axis of x are

$$\frac{c^2}{r}\sqrt{\frac{a^2-r^2}{a^2-c^2}}, \quad \text{and} \quad \frac{b^2}{r}\sqrt{\frac{a^2-r^2}{a^2-b^2}},^*$$

the latter of which, when b and consequently r very nearly equals a, becomes

$$\frac{b^2}{r}\sqrt{1-\frac{r^2-b^2}{a^2}} \text{ very nearly,}$$

which may be made to approach b as nearly as we please by diminishing the difference between r and b, that is by throwing the pole very near to the boundary of the elliptic section mentioned in the text.

The other semi-axis will be increased by this means; but very slightly, as a^2-r^2 is supposed to be very small in comparison with a^2-c^2.

* Note (19). The equation to the projection on yz is obtained from that on xy by writing z for x, a for c, and c for a.

In bodies where one of the extreme moments of inertia differs little from the mean moment, and where consequently the central ellipsoid is nearly a solid of revolution about the other axis, the stability of the rotatory motion is only absolute for this axis. This is the case in the Earth, whose motion is stable about its present axis, but would be very much otherwise about the third axis, which differs, as we know, very little from the mean axis.

Motion of the principal Axes of the Body in absolute Space.

We have considered the motion of the instantaneous pole of rotation both in the body and in space: but it may be asked what are the motions of the poles of the central ellipsoid themselves; the velocities with which they revolve about the fixed axis of the impressed couple, and with which they approach or recede from the plane perpendicular to this axis which gives us their motions of *Precession* and *Nutation:* we may examine into the nature of the three curves or serpoloids which the projections of these three principal poles trace at the same time on a fixed plane, &c. and we shall find in the easiest manner many curious properties of the motion of a body. For example,

The sum of the areas swept out, during the motion of the body, by the projections on the plane of the couple of equal portions of each of the three

principal axes, measured from the centre, is proportional to the time.

If these three lines, instead of being equal, are proportional to the square roots of the moments of inertia, or to what I call the arms of inertia of the body about the same axes, the sum of the areas is also proportional to the time.

These are simple, and in some degree geometrical theorems, which must however be distinguished from the dynamical theorem relative to the areas traced by all the radii vectores drawn to the several molecules of the body, though it is easy to reduce these expressions, drawn from the same principle, to one and the same.

Analogous theorems may be proved relative to the *nutations* of the three principal axes of the body towards the fixed plane of the couple of impulsion. For the sum of the squares of the distances of the three principal poles of the central ellipsoid from the axis of the couple is a *constant quantity:* and,

The sum of the squares respectively multiplied by the moments of inertia of the body is constant throughout the whole motion.

Lastly, if we consider the curves described by the projections of these poles on the same fixed plane, we shall find that they are of the same nature with the serpoloid described by the instantaneous pole of rotation.

In general the pole, whether it be the *greater* or *lesser*, which forms the centre of the poloid, describes a curve having like the serpoloid equal

and regular undulations about the same centre; the superior vertices of the one corresponding to the superior vertices of the other, and the inferior to the inferior. During the same time the other two poles describe also curves undulating regularly: but when one of them passes the superior vertices of its path the other is passing the inferior vertices of its similar path; the poles being at an angular distance of 90° from one another.

In the particular cases in which the *poloid* is an *ellipse* and the *serpoloid* a spiral, the mean pole of the body describes also a spiral which approaches the centre continually, and nearer than by any assignable limit, without ever reaching it: the two other poles also describe spirals, of a species in some degree resembling the former; for each of them recedes continually from a certain *minimum* distance from the centre to a certain *maximum* which it never reaches; so as to approach continually the circumference of an asymptotic circle. We may make another curious remark on this particular case of the motion of bodies; viz. that there exists in the mean plane of the central ellipsoid a certain diameter which has the remarkable property of remaining always perpendicular to the fixed axis of the impressed couple, and therefore of describing the plane of this couple, and that too with an uniform motion. So that *the whole motion of the body consists in turning on this particular diameter with a variable velocity, while this diameter uniformly describes a circle in space.*

When the central ellipsoid is one of revolution the pole of the figure describes a circle as well as the instantaneous pole. In this case there is no other pole properly speaking than the extremity of the axis of the spheroid: but if we chose to fix arbitrarily on two other points in the equator at an angular distance of 90^0 from each other and to examine the two curves which their projections describe, we should have two perfectly equal but not circular curves: which would be two equal serpoloids about the same centre, the superior vertices of the one being always at an angular distance of 90^0 from the corresponding inferior vertices of the other.

We shall have yet more new properties and new illustrations of rotatory motion to present. For instance, it is easy to see that *the section of the central ellipsoid made by the fixed plane of the couple is an ellipse whose area is constant throughout the motion.*

So that if we consider the central ellipsoid as plunged into a non-resisting fluid, of which the fixed plane of the couple forms the level, we may say that the area of the plane of floatation is constant.

We may pass from hence to a new illustration of the motion and represent it by that of an elliptical cone which rolls on the plane of the couple with a variable velocity, and slides with an uniform velocity. All which properties will be developed in the Memoir.

We see how much these illustrations enlighten and correct our ideas of even the most elementary portions of the theory of rotatory motion. Those who cultivate the geometrical properties of surfaces of the second order will draw from them without difficulty a great number of curious theorems relative to this kind of motion: for each proposition in Geometry gives a corresponding one in Dynamics. But the great advantage of our mode of treating the subject consists in the easy demonstrations which it affords of the motions of precession and nutation of the equators of the heavenly bodies, and of the nodes of their orbits: of thus simplifying and sometimes correcting these difficult theories, of which we have already seen an example in the determination of the *single* and *invariable* plane of areas, which I have denominated the *Equator of the system of the Universe.*

APPENDIX.

I. *The Axis of Instantaneous Rotation.*

When a rigid body is in motion, it is turning, during every separate instant, about some straight line or other. (v. page 17.)

For an analytical proof of the existence of this line see Whewell's *Dynamics*. (Art. 120.)

The following is extracted, by permission of the Author, from Earnshaw's *Statics*. (Art. 109.)

" Let P, Q, (fig. 18.) be any two particles of a rigid body; PP', QQ', the paths which they describe during the same instant; A, B, the centres of curvatures of these paths; then the line joining A, B, will be the axis about which the whole body turns during this instant."

" For the lines which join the successive contempo raneous positions of P and Q, while they are respectively passing to P' and Q', will form a species of conical surface; and since, by reason of the rigidity of the body, they are all of the same length, the planes PAP', QBQ', in which the curves (PP', QQ') formed by their extremities lie, must be parallel. Now since P describes round A the angle PAP' in the same time that Q describes round B the angle QBQ', the trapezium $PABQ$ turns in the same time round AB and comes into the position $P'ABQ'$ (for $AP = AP'$, and $BQ = BQ'$, since A, B, are the centres of curvature of PP', QQ'). Con-

sequently the motion of every particle of the body situated in PQ takes place about the axis AB."

"Hence the motions of the points P and Q take place about AB, and therefore AB must be perpendicular to PAP', QBQ', the planes of these motions. In like manner if the motion of any other particle R take place about a point C, AC must be perpendicular to the planes of motion PAP', RCR'; hence both AB and AC are perpendicular to PAP', which is impossible, (Euc. XI. 13.) unless they coincide; in which case C is a point in AB, and the motion of R takes place about AB; and since R is any particle, the motion of every particle takes place about AB, that is, the whole body turns during the instant about the straight line AB."

If any point in the body be fixed the axis must pass through this point.

For let O be the point and join AO; then we may consider PAO as a crooked but rigid rod moveable about the fixed point A, and it is manifest that while one extremity P moves through PP', the other cannot remain at rest unless it lies in AB.

In this case the motion of any one point P determines the motion of every other point.

For if PO be joined, the rod PO considered as a rigid body must be turning during every separate instant about some axis passing through O; and it is shewn in the course of the above demonstration that the plane of the motion of any other point Q in the rigid body is parallel to the plane of the motion of any point in OP. Therefore the motion of Q is about the same axis as that of OP; and it is clear from note (1) that the angular velocities are the same.

If the body is perfectly free, it has during every instant a simple rotatory motion about some axis pass-

ing through the centre of gravity; except in the case when all the particles move in equal and parallel straight lines, that is, when the body has a mere motion of translation.

To prove this it is necessary to establish the following Dynamical property of the centre of gravity.

If a motion of translation be communicated to a body which has a simple rotatory motion about any axis passing through its centre of gravity, the motions will subsist together, and each will continue to affect the body precisely as it would have done if the other had never existed.

Suppose a velocity v in the direction of a line which makes angles α, β, γ, with the co-ordinate axes to be communicated to every particle of the rigid system in note (11).

Then the resolved parts of the effective forces which act on a particle M situated at the point P are

$$\begin{aligned} -m\omega y + mv\cos\alpha, &\text{ in the direction } Gx, \\ m(\omega x + v\cos\beta) \,\ldots\ldots\ldots\ldots\ldots\ldots &\; Gy, \\ mv\cos\gamma \,\ldots\ldots\ldots\ldots\ldots\ldots &\; Gz. \end{aligned}$$

And the resolved parts of all the elementary effective forces are reducible to

(i.) Three forces applied at G; viz.

$$-\omega\Sigma(my) + v\cos\alpha\,\Sigma(m),$$

which by the property of the centre of gravity (if $\mu = \Sigma(m)$ the mass of the system)

$$\begin{aligned} &= \mu v\cos\alpha, \text{ in the direction } Gx, \\ \omega\Sigma(mx) + v\cos\beta\,\Sigma(m) &= \mu v\cos\beta \,\ldots\ldots\ldots\ldots\ldots\ldots\; Gy. \\ v\cos\gamma\,\Sigma(m) &= \mu v\cos\gamma \,\ldots\ldots\ldots\ldots\ldots\ldots\; Gz. \end{aligned}$$

(ii.) Two couples; viz.

$$\Sigma m z\,(-\,\omega y + v\cos\alpha) - \Sigma m x v \cos\gamma$$

$$= -\,\omega\Sigma m y z \text{ in the plane } zx,$$

$$\Sigma m z\,(\omega x + v\cos\beta) - \Sigma m y v \cos\gamma$$

$$= \omega\Sigma m x z \,\ldots\ldots\ldots\ldots\ldots\ldots\, zy.$$

(iii.) Two couples; viz:

$$\Sigma m x\,(\omega x + v\cos\beta) \text{ and } \Sigma m y\,(\omega y - v\cos\alpha),$$

which together $= \omega\Sigma m\,(x^2 + y^2)$ in the plane xy.

The whole elementary effective forces are therefore equivalent to a force equal to the resultant of the forces (i) $= \sqrt{\mu^2 v^2\,(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)} = \mu v$ applied at the centre of gravity, in a direction making angles α, β, γ, with the axes, and to the resultant of the two sets of couples (ii) and (iii); and it is manifest that if the rotatory motion were suppressed, the resultant force would still be μv, and the resultant couple would vanish; and therefore conversely that the consequence of applying a force $= \mu v$ at the centre of gravity of the body is to give every particle of it a mere motion of translation with a velocity v, in the direction of the force, whether the body has or has not a rotatory motion; and that on the other hand the couples (ii) and (iii), which are necessary and sufficient to produce a simple rotatory motion about an axis through G, remain the same, whatever be the value of v, and therefore of μv.

Suppose now that AB (fig. 19.) the line containing the centres of curvature of the paths of every particle of the body *in absolute space*, does not pass through the centre of gravity G of the body, and that the angular velocity of the body about it is ω. Then if about Gg a line through G parallel to AB, we suppose two angular

velocities each $= \omega$ to be communicated to the body in opposite directions, we shall have a simple rotatory motion with an angular velocity ω about Gg and a motion of translation perpendicular to the plane $GgAB$; and since Gg passes through the centre of gravity these motions are independent, that is, the body has during the instant, a *simple* rotatory motion about an axis through G.

Hence also it is impossible for a free rigid body to have a simple rotatory motion about any axis which does not pass through the centre of gravity.

For the rotatory motion which it has for an instant about any such axis can always be resolved into a simple one about an axis through the centre and a motion of translation. Hence the axis of the screw spoken of in pp. 24, 25, always passes through the centre of gravity of the body.

We may also illustrate the above principles by a reference to the motions of the Earth and Moon in absolute space.

If the Earth had no rotatory motion about its own axis we might naturally consider SK (fig. 20.) perpendicular to the plane of the ecliptic as the instantaneous axis (AB figs. 18 and 19.). The motion about it would be equivalent at every instant to a motion of translation in the direction of a tangent to the Earth's orbit, and a simple rotatory motion about an axis through the centre of gravity of the Earth parallel to SK. In the course of a year the Earth would turn once round on this axis in the direction WLE, and to a spectator on the Earth's surface the Sun would appear to describe in the *same* direction a circle round the centre of gravity of the Earth in the plane of the ecliptic, and the fixed stars parallel circles in the *opposite* direction. And this motion would be totally independent of the motion

of translation, of which the spectator would only be aware from the consideration that if it were suppressed the Sun's apparent motion would be in the *opposite* direction.

But no such apparent annual motion of the fixed stars is observed. We may therefore conclude that the Earth has no such simple rotatory motion, and that its motion round the Sun is a simple motion of translation of the centre of gravity; which would also appear from Dynamical considerations. It is observed that the Moon presents always the same face towards the Earth. Hence all the particles in it describe similar curves about a line through the centre of gravity of the Earth perpendicular to the Moon's orbit. This axis is therefore always an instantaneous axis of the Moon. Hence the Moon has a simple rotatory motion in the same direction with its motion of translation about an axis through its centre of gravity parallel to the instantaneous axis, and turns once round on this axis during a revolution of its centre of gravity round that of the Earth.

If now we suppose one or more rotatory motions to be communicated about other axes through the centre of gravity, the motion of translation at every instant will not be affected, and the simple rotatory motion about the centre of gravity at every instant, upon which the appearances of the heavens depend, will be compounded of these.

II. *Principal Axes and Moments of Inertia.*

Through every point in a material system at least three straight lines may be drawn, in directions mutually at right angles, for which, when they are severally taken as the axis of z, each of the quantities $\Sigma(mxz)$, $\Sigma(myz)$ vanishes.

In a rigid body or system these lines are called principal axes. (v. page 28.)

If GP (fig. 11.) make angles α, β, γ, with the co-ordinate axes, and the co-ordinates of Q, the position of any particle m, be x, y, z; as in note (13).

$$\text{Moment round } GP\ (P) = \Sigma m\,(QR)^2$$

$$= \Sigma m\,(x^2 + y^2 + z^2 - GR^2)$$

$$= \Sigma m x^2 + \Sigma m y^2 + \Sigma m z^2$$

$$- \Sigma m x^2 \cos^2 \alpha - \Sigma m y^2 \cos^2 \beta - \Sigma m z^2 \cos^2 \gamma$$

$$- 2\Sigma m x y \cos \alpha \cos \beta - 2\Sigma m x z \cos \alpha \cos \gamma$$

$$- 2\Sigma m y z \cos \beta \cos \gamma.$$

Let $A = \Sigma m y^2 + \Sigma m z^2$, which is manifestly the moment round Gx,

$$A' = \Sigma m y z, \text{ and so on for the other axes.}$$

$$\text{Then} \quad P = A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma$$

$$- A' \cos \beta \cos \gamma - B' \cos \alpha \cos \gamma - C' \cos \alpha \cos \beta.$$

And if we take any point x, y, z in GP at a distance p from G, we shall have

$$Pp^2 = Ax^2 + By^2 + Cz^2 - A'yz - B'xz - C'xy.$$

Now the value of P evidently depends upon the values of α, β, γ, that is upon the position of GP; and p being arbitrary we may take it equal to any function of the same quantities;

$$\text{let it} = \frac{1}{\sqrt{n \,.\, P}};$$

$$\therefore \ Pp^2 = \frac{1}{n},$$

and $Ax^2 + By^2 + Cz^2 - A'yz - B'xz - C'xy = \frac{1}{n} = c,$

is the equation to the locus of the extremity of p. We see at once that it is a surface of the second order whose centre is G; and since P can never vanish, except in the particular case when all the particles lie in the straight line GP, p is always finite, and the surface is an ellipsoid.

If now the axes of the co-ordinates be transformed so as to coincide with the Geometrical Axes of the ellipsoid, the form of the equation becomes

$$Ax^2 + By^2 + Cz^2 = c;$$

For these axes therefore the quantities A', B', C', respectively $= 0$; and therefore each of them is a principal axis of the body or system.

Now a geometrical axis of an ellipsoid coincides with the normal at the point where it meets the surface; at this point therefore we have

$$\frac{Ax - C'y - B'z}{x} = \frac{By - A'z - C'x}{y} = \frac{Cz - B'x - A'y}{z},$$

$$\text{or } \frac{A \cos \alpha - C' \cos \beta - B' \cos \gamma}{\cos \alpha}$$

$$= \frac{B \cos \beta - A' \cos \gamma - C' \cos \alpha}{\cos \beta}$$

$$= \frac{C \cos \gamma - B' \cos \alpha - A' \cos \beta}{\cos \gamma} = P;$$

whence we obtain

$$(P - A) \cos \alpha + C' \cos \beta + B' \cos \gamma = 0,$$

$$C' \cos \alpha + (P - B) \cos \beta + A' \cos \gamma = 0,$$

$$B' \cos \alpha + A' \cos \beta + (P - C) \cos \gamma = 0.$$

and by elimination,

$$(P-A)(P-B)(P-C)-A'^2(P-A)-B'^2(P-B)-C'^2(P-C)$$
$$+2A'B'C'=0,$$

in which the roots of P are of course the moments of inertia about the geometrical axes of the ellipsoid.

To shew that the roots of this equation are real, assume $A'=0$, $B'=0$; then the factor $P-C$ disappears and the equation becomes a quadratic, and if this has for its roots P_1, P_2, which we find to be possible quantities, and we substitute successively in the original equation

$$-\infty, \; P_1 \; P_2, \text{ and } +\infty,$$

we obtain results alternately positive and negative, whence we conclude that there are three real roots between these limits, (vide *Cauchy*, *Exercices*, Vol. III. p. 5.)

The above ellipsoid is manifestly the same with that which Poinsot calls the central ellipsoid.

The moments of inertia about the principal axes passing through the centre of gravity are called the principal moments of inertia of the body.

The moment about the major axis of the ellipsoid is clearly less, and that about the minor axis greater, than that about any other axis whatever.

If two of these are equal the ellipsoid whose equation we have found above becomes a spheroid, and every diameter in the equatoreal plane is a principal axis, or the number of principal axes is infinite.

If the three principal moments are equal, the ellipsoid becomes a sphere, and every diameter is a principal axis.

When the co-ordinate axes are the principal axes through the centre of gravity, the value of P is reduced to

$$A \cos^2 \alpha + B \cos^2 \beta + C \cos^2 \gamma.$$

From this we can deduce the moment about any axis which does not pass through the centre.

For if a rigid body or system revolve about a fixed axis AB (fig. 19.) with an angular velocity ω, the resultant effective couple in the plane Pag perpendicular to $AB = \omega \times$ (moment of inertia (A) round AB).

But if the axis were not fixed the motion might be resolved as before into a rotatory motion round Gg, and a motion of translation; the former of which gives us a couple $= \omega \times$ (moment (G) round Gg), and the latter a force applied at G

$$= \mu \text{ (velocity of translation)}$$

$$= \mu (\omega \cdot a) \text{ if } GO = a, \text{ see note (5),}$$

which is equivalent to a force $\mu\omega a$ applied at O and a couple $= \mu\omega a \cdot a$ in a plane perpendicular to GOA.

And the force is destroyed by the resistance of the fixed axis;

$$\therefore \; \omega A = \omega G + \omega \mu a^2,$$

$$A = G + \mu a^2.$$

The principal moments of a homogeneous solid body are readily determined by integration.

For if $\mu = \Sigma (m)$ be a continuous function of x and y, the value of an individual elementary portion $\Delta \mu = m$ of it must entirely depend on the values of x and y at the point where that portion is situated.

Therefore the corresponding elementary portion ΔC of the moment, which $= \Delta\mu \,.\, (x^2 + y^2)$, must depend entirely on the values of x and y, and must therefore be a function of μ.

$$\text{But } \frac{\Delta C}{\Delta\mu} = x^2 + y^2;$$

$$\therefore \text{ taking the limits, } d_\mu C = x^2 + y^2.$$

$$\text{Similarly } d_\mu A = y^2 + z^2,$$

$$d_\mu B = x^2 + z^2.$$

If the density of the body vary according to a given law of the position of a particle,

$$d_x\, d_y\, d_z \mu = \rho \times f(x, y, z);$$

$$\therefore\ d_z \mu = \rho \int_x \int_y f(x, y, z),$$

$$\text{and } d_z C = \rho \int_x \int_y (x^2 + y^2) \times f(x, y, z)$$

and similarly for A and B.

If the body is homogeneous,

$$d_x\, d_y\, d_z \mu = \rho; \quad \therefore\ d_z \mu = \rho \int_x \int_y 1,$$

$$\therefore\ d_z C = \rho \int_x \int_y (x^2 + y^2) = \rho \int_x \int_y x^2 + \rho \int_x \int_y y^2,$$

and similarly for A and B.

For a plane surface $z = 0$; and $d_x\, d_y\, \mu = \rho$;

$$\therefore\ C = \rho \int_x \int_y x^2 + \rho \int_x \int_y y^2,$$

$$A = \rho \int_x \int_y y^2, \qquad B = \rho \int_x \int_y x^2;$$

$$\therefore\ C = A + B.$$

If AB (fig. 21.) be an uniform physical line whose middle point is G, which is clearly the centre of gravity, and a line Gg perpendicular to it be taken for the axis of (z), we shall evidently have $z = 0$ for every point in this line:

$$\therefore\ \Sigma(mxz) = 0, \quad \Sigma(myz) = 0,$$

and every line perpendicular to AB is a principal axis thereof.

To find the principal moment, take GA for the axis of x, then $y = 0$ and $d_x\mu = \rho$;

$$\therefore\ C = \rho \int_x x^2$$

$$= \rho \frac{x^3}{3} + C,$$

which from $x = -\frac{a}{2}$ to $x = +\frac{a}{2}$

$$= \rho \cdot \tfrac{1}{3} \cdot \frac{a^3}{4}$$

$$= \mu \cdot \frac{a^2}{12}.$$

The moment about an axis through A parallel to Gg

$$= \mu \cdot \frac{a^2}{12} + \mu \cdot \frac{a^2}{4}$$

$$= \mu \cdot \frac{a^2}{3}.$$

The moment about an axis Gh inclined at an angle (α) to Gg

$$= \mu \cdot \frac{a^2}{12} \cos^2 \alpha.$$

If $ABCD$ (fig. 22.) be a rectangular parallelogram, any axis perpendicular to its plane will be a principal axis; and lines through G parallel to the sides will evidently be principal axes at G, since for every product $+ mxy$ we shall have a corresponding product $- mxy$,

and $\therefore$ $\Sigma mxy = 0$, and if $AD = a$, $AB = b$, $GN = y$,

$$d_y \mu = \rho \, . \, MM' = \rho \, . \, AD;$$

$$\therefore \text{ moment round } Gx\,(A) = \int_y y^2 \, . \, \rho a$$

$$= \rho a \frac{b^3}{12} = \mu \frac{b^2}{12},$$

$$B = \rho b \frac{a^3}{12} = \mu \frac{a^2}{12}.$$

Hence in this case

$$C = \mu \, . \frac{a^2 + b^2}{12},$$

and if we suppose this axis to pass through the centre of gravity of any number of rectangles of the same size and density, and exactly parallel to $ABCD$, the moment of the system about this axis will evidently be obtained by multiplying C by the number of these planes. Hence the principal moments of a rectangular parallelopiped whose base is $ABCD$ will be

$$C = \rho abc \, . \frac{a^2 + b^2}{12} = \mu \frac{a^2 + b^2}{12}, \text{ and similarly,}$$

$$B = \mu \, . \frac{a^2 + c^2}{12}, \quad A = \mu \, . \frac{b^2 + c^2}{12}.$$

Whenever a homogeneous solid body can be divided by three planes, passing through the centre of gravity at right angles to each other, into perfectly symmetrical portions, the intersections of these planes are principal axes. This follows readily from what has just been said of the principal axes of a rectangle; it appears also from the consideration that each plane will in that case contain two of the geometrical axes of the central ellipsoid.

If APB be an ellipse (fig. 23.), the principal axes of G are GA, GB, and a line perpendicular to the plane APB,

$$\text{and } B = \rho \textstyle\int_x x^2 . PP', \quad \text{since } d_x \mu = PP',$$

$$= 2\rho \frac{b}{a} \textstyle\int_x x^2 \sqrt{a^2 - x^2}$$

$$= 2\rho \frac{b}{a} \left\{ -\tfrac{1}{3} (a^2 - x^2)^{\frac{3}{2}} . x + \tfrac{1}{3} \textstyle\int_x (a^2 - x^2)^{\frac{3}{2}} \right\},$$

$$\tfrac{1}{3} B = 2\rho \frac{b}{a} \left\{ \tfrac{1}{3} \textstyle\int_x x^2 \sqrt{a^2 - x^2} \right\};$$

$$\therefore \tfrac{4}{3} B = 2\rho \frac{b}{a} \left\{ \tfrac{1}{3} a^2 \textstyle\int_x \sqrt{a^2 - x^2} - \tfrac{1}{3} x (a^2 - x^2)^{\frac{3}{2}} \right\};$$

$$\therefore B = \tfrac{1}{2} \rho \frac{b}{a} \left\{ a^2 . \left(\text{circular area} \sin^{-1} \frac{x}{a} \right) - x (a^2 - x^2)^{\frac{3}{2}} \right\} + C,$$

which from $x = -a$ to $x = +a$

$$= \tfrac{1}{2} \rho \frac{b}{a} . a^2 . \frac{\pi a^2}{2} = \rho \frac{\pi a^3 b}{4}$$

$$= \mu \frac{a^2}{4},$$

$$\text{and } A = \mu \frac{b^2}{4}; \qquad \therefore C = \mu \frac{a^2 + b^2}{4}.$$

The principal moments for a circle are immediately deducible by making $b = a$, as are those for an elliptic or circular cylinder by multiplying by the height.

We may however obtain C more readily for a circle.

$$\text{For if } GR \text{ (fig. 24.)} = r,$$

$$r^2 = x^2 + y^2 = d_\mu C, \text{ and } d_r \mu = 2\pi r \, . \, \rho;$$

$$\therefore \; C = 2\pi\rho \int_r r^3 = \rho\frac{\pi a^4}{2} = \mu\frac{a^2}{2}, \text{ if } GA = a.$$

Every radius GA is a principal axis;

$$\therefore \; A = B = \frac{C}{2} = \mu\frac{a^2}{4}.$$

For a solid of revolution about the axis of z,

$$d_z \mu = \pi r^2 \, . \, \rho = \pi (fz)^2 . \, \rho;$$

$$\therefore \; C = \pi\rho \int_z (fz)^4.$$

$$\text{And } A = B;$$

$$\therefore \; d_\mu A = \tfrac{1}{2}(d_\mu A + d_\mu B) = \tfrac{1}{2}(x^2 + y^2) + z^2$$

$$= \tfrac{1}{2} r^2 + z^2;$$

$$\therefore \; A = \frac{\pi\rho}{2}\int_z (fz)^4 + \pi\rho\int_z (fz)^2 . \, z^2.$$

If G be the centre of an ellipsoid (fig. 25.) the axes GA, GB, GC are principal axes;

$$\text{and } d_x A = \rho \int_y \int_z (y^2 + z^2).$$

But if PNP' be a section at the distance $GM = x$, the equation to it is

$$\frac{y^2}{b^2} \cdot \frac{1}{1 - \frac{x^2}{a^2}} + \frac{z^2}{c^2} \cdot \frac{1}{1 - \frac{x^2}{a^2}} = 1;$$

$$\therefore\ MN = b\sqrt{1 - \frac{x^2}{a^2}}, \quad MP = c\sqrt{1 - \frac{x^2}{a^2}},$$

and $\rho \int_y \int_z (y^2 + z^2)$ between the limits

$$y = -MN, \qquad z = -MP,$$

$$y = +MN, \qquad z = +MP,$$

is manifestly the moment of inertia of the plane ellipse PNP' about GA;

$$\therefore\ d_x A = \frac{\pi\rho}{4} bc \cdot \left(1 - \frac{x^2}{a^2}\right) \left\{b^2 \left(1 - \frac{x^2}{a^2}\right) + c^2 \left(1 - \frac{x^2}{a^2}\right)\right\};$$

$$\therefore\ A = \frac{\pi\rho}{4} \cdot bc\,(b^2 + c^2) \left\{x - \frac{2}{3}\frac{x^3}{a^2} + \frac{1}{5}\frac{x^5}{a^4}\right\} + C,$$

which from $x = -a$ to $x = +a$

$$= \frac{4\pi\rho}{15} abc \cdot (b^2 + c^2) = \mu \cdot \frac{b^2 + c^2}{5}.$$

In all the above cases the moment of inertia is of the form μk^2, where k is a constant quantity. And this will be the case in any system whatever, since the moment is made up of positive products $m\,(x^2 + y^2)$ each of which is of that form.

The line k is called *the radius of gyration.*

If G (fig. 26.) be the centre of gravity of a free rigid system, k the radius of gyration at G, and Gg in the plane of the paper a principal axis perpendicular to GO, an impulsive force S applied in a direction perpendicular to the plane of the paper at a point G whose distance from G is s, is equivalent to a force S applied at G and a couple Ss in a plane perpendicular to Gg.

The effect of the former is to produce a motion of translation in the direction of S. Let v be the velocity generated, then $S = \mu v$.

The effect of the latter is to produce a rotatory motion round GC. Let ω be the velocity of rotation,

$$\text{then } S \cdot s = \omega \mu k^2.$$

Let C be a point such that $GC \cdot \omega = v$, then C and every point in a line through C parallel to Gg will remain at rest; therefore this line will be the axis about which the body or system will actually turn during the first instant *in absolute space.*

The line thus obtained by reversing the process in pp. 76, 77, and compounding the motion of translation with the rotatory motion is called the *Spontaneous Axis of Rotation* (v. p. 25.), which is therefore a principal axis at a point C which is sometimes called the *Centre of Spontaneous Rotation*, and whose position in OG produced is determined by the consideration that

$$GC \cdot \frac{S \cdot s}{\mu k^2} = \frac{S}{\mu}; \qquad \therefore \; GC = \frac{k^2}{s}.$$

If the axis Cc were fixed, the shock of any force S on O would evidently produce no pressure whatever on Cc. Hence O is called the *Centre of Percussion* corresponding to the axis Cc. Its position is determined from the equation

$$GO = \frac{k^2}{CG}.$$

If a rigid body or system turn about any fixed axis Cc with an angular velocity ω, the motion at any instant is equivalent to a simple rotatory motion about an axis Gg through the centre of gravity parallel to Cc, and a couple of rotatory motions whose moment is $CG . \omega$, that is, a motion of translation with a velocity $= CG . \omega$; for these are the motions into which, if the axis were set free, it would immediately be resolved.

Now the motion of G is unaffected by the former, and the latter would be generated by a force S applied at $G = \mu . CG . \omega$.

Therefore the resultant couple of the effective forces in a plane perpendicular to Cc

$$= \omega . (\text{moment of inertia round } Cc)$$

$$= \frac{S}{\mu . CG} (\mu k^2 + \mu CG^2) \text{ (page 83.)}$$

which, if O be a point such that $GO = \dfrac{k^2}{CG}$, $= S . CO$.

But a force S applied at O is equivalent to a force S at C, which would be destroyed by the reaction of Cc, and a couple whose moment is $S . CO$.

Hence a rotatory motion about a fixed axis Cc with a velocity ω would be produced by a force $S = \omega . \mu . CG$ applied at O.

But since the axis is fixed, the same effect would be produced by a force R applied at G

$$= S . \frac{CO}{CG} = \omega . \mu . CO.$$

Now if the whole mass were collected at O and connected with C by an imponderable rigid rod, the force which must be applied at O, which would then become the centre of gravity, to cause μ to revolve about C would be

$$\mu \times (\text{linear velocity}) = \mu . \omega CO, \text{ as before.}$$

It is evident that the velocity of the particle m at O is the same in either case.

Any point O lying in a cylindrical surface at a distance $CG + \dfrac{k^2}{CG}$ from Cc is called a *centre of oscillation* corresponding to the axis Cc.

The angular velocity due to a force R at $G = \frac{R}{\mu \, . \, CO}$, and the circumstance of the axis being fixed enables us to replace any force T acting at Q perpendicular to the plane of the paper by a force $R = T \, . \, \frac{CQ}{CG}$ at G, parallel to T.

The same reasoning holds for a succession of impulsive forces, and thus for a continued force, such as gravity.

If CA (fig. 27.) be a vertical, $ACG = \theta$, the impulsive force at G, which acts at every successive instant during the short time Δt to produce an angular velocity $\Delta \omega$,

$$= \mu g \sin \theta.$$

And each successive increment of velocity being independent of the former, the whole increment $\Delta \omega$ in the time Δt

$$= \Delta t \, . \, \frac{\mu g \sin \theta}{\mu \, . \, CO};$$

$$\therefore \frac{\Delta \omega}{\Delta t} = \frac{g \sin \theta}{CO};$$

$$\therefore \text{ taking the limits } d_t \omega = \frac{g \sin \theta}{CO}.$$

But $\Delta \theta$, the angle described in any small time Δt, during which ω may be considered uniform,

$$= \omega \, . \, \Delta t;$$

$$\therefore d_t \theta = \omega;$$

$$\therefore d_t^2 \theta = d_t \omega = \frac{g \sin \theta}{CO}$$

The same expression for determining t which would have arisen in considering the motion of any mass suspended from C by an immaterial thread CO.

III. *Conservation of Couples. Equations of Euler.*

The following statement and demonstration of the principle of the Conservation of Couples are translated from the Memoir presented by M. Poinsot to the Institute in May 1804.

Let any number of perfectly free bodies, unconnected with each other, describe uniformly straight lines in space; then the forces by which they are respectively impelled remain always the same and in the same direction.

Therefore the resultant of all these individual forces passing through any fixed point, and the resultant couple, are the same at every instant during the whole motion.

Now if we suppose these bodies to be suddenly connected together so as to act one upon another by virtue of any reciprocal forces whatever, that is to say, such that between any two bodies the action and reaction are perfectly equal and opposite, which comprehends all forces of this nature, the individual motions of the several bodies will be changed, and the forces which respectively impel them will vary in magnitude and direction during every instant of the motion. But the resultant of these forces passing through any fixed point and the resultant couple will remain the same as before, and would still be the same if the bodies were suddenly set free, and each were to fly off in a straight line with the new velocity by which it is actually impelled.

This principle, which follows from the Differential Equations of Motion, may also be demonstrated in the following manner.

Since each body by reason of its connection with the others is unable any longer to obey fully the impulse which it has received, the force which acts on it is decomposed into two others, one of which is destroyed, while the other is that which the body actually obeys. The circumstances are identical with those of the case in which a body in motion meets with an insurmountable obstacle, except that the component part of the force which would then be annihilated, proceeds to act upon the other bodies of the system, and is destroyed by the united action of the similar components which each of them contributes.

Now it is evident that the resultant force passing through any fixed point, and the resultant couple, of the impulsive forces are respectively equivalent to the resultant forces and resultant couples of the two sets of forces into which they are decomposed. But the first set being in equilibrio their resultant force and resultant couple vanish by the laws of Statics: therefore the original resultant force and resultant couple are identical with those which the body actually obeys. With respect to any other forces, such as mutual attractions, which may exist in the system, since they are reciprocal, that is to say, distributed in pairs of equal and opposite forces, they cannot in any way affect the forces above mentioned.

We see therefore that in a system of bodies which have received any primitive impulses and which act mutually upon each other in any manner, the resultant of all the forces which impel them passing through any fixed point, and the resultant couple, remain always the same whatever variations the respective moving forces of individual bodies may experience, whether these variations take place by insensible degrees, or abruptly, from any change in the reciprocal actions of the bodies, or from the sudden introduction of any new connecting forces among them.

Hence in note (16) the resultant of all the effective couples at any instant during the motion = M.

The centrifugal couple is of course included. It is the portion of the resultant $M \sin \theta'$ of the couples whose planes pass through GP, which lies in a plane passing also through GM.

The projections of $GM = M$ (fig. 12.) on three fixed lines Gx, Gy, Gz in space, will represent in magnitude and direction the axes of the resultant couples of the effective forces in planes perpendicular to these lines.

Let a, b, c, be the cosines of the angles which the axes of the ellipsoid make with the line Gx, then

$$M \cos MGx = M \cos \alpha \,.\, a + M \cos \beta \,.\, b + M \cos \gamma \,.\, c$$
$$= A\omega_a \,.\, a + B\omega_b \,.\, b + C\omega_c \,.\, c.$$

Similarly

$$M \cos MGy = A\omega_a \,.\, a' + B\omega_b \,.\, b' + C\omega_c \,.\, c',$$

$$M \cos MGz = A\omega_a \,.\, a'' + B\omega_b \,.\, b'' + C\omega_c \,.\, c''.$$

(v. Poisson, *Mécanique*, Art. 409.)

It must be observed that the portion of the centrifugal couple, as obtained in note (15), which lies in the plane perpendicular to the axis of instantaneous rotation is omitted in deducing the final equation, as having no tendency to change the position of the axis.

IV. *Application to the Precession of the Equinoxes.*

Let G (fig. 28.) be the Earth's centre, GC its geometrical axis; S the Sun; $S'LW$ the equator, the Earth being supposed in the position which it has at the

summer solstice. Then the action of the Sun on S' is greater, and on W less than the action on G. Therefore in addition to the force on G which produces the motion of translation there is a couple of forces in opposite directions which produces a rotatory motion round the line ♈ L perpendicular to the plane KGS. Also in winter the opposite face being presented to S the couple tends to produce motion in the same direction. If therefore the Earth when originally projected had a rotatory motion of its own about GC, which would be the case if the primitive impulse did not pass through G but through some other point in the equatoreal plane in a direction perpendicular to a plane passing through GC, these two rotatory motions would be compounded, and the pole of rotation on the central ellipsoid would be drawn aside to a short distance from the geometrical pole. It would therefore describe as its poloid a circle at this distance from the pole C. The serpoloid would also be a circle about GK. For the plane of the resultant couple would be parallel to the plane of the ecliptic.

The ellipticity of the Earth being very small the effect of the centrifugal forces is not perceptible.

When the Earth is in any other position the effect of the Sun's attraction is at some times to increase and at others to diminish the obliquity, which however it does not permanently alter.

To determine the velocity of the Pole we must know the amount of the effect of the disturbing couple at S' and W, which depends on the Sun's attraction at those points. (v. Airy, *Precession*, Art. 21.)

If a be the angular velocity generated by the couple at S' and W, ω the angular velocity of the Earth's rota-

tion round the instantaneous axis which is always the same, ρ the radius of the poloid,

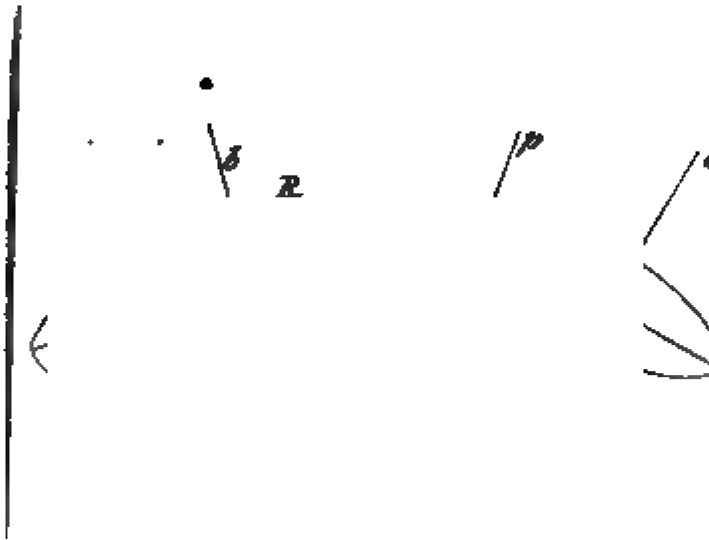
$$2\pi\rho = (\text{velocity of the Pole}) \times (\text{one day}).$$

$$\text{But by note (26.), vel. of the Pole} = (\text{radius of } \oplus)\,\frac{a}{\omega},$$

$$\text{and length of a day} = \frac{2\pi}{\omega};$$

$$\therefore\ \rho = (\text{radius of } \oplus)\,.\,\frac{a}{\omega^2}. \quad \text{(v. Airy, Art. 13.)}$$

The value of ρ is obtained by observation in note (9).



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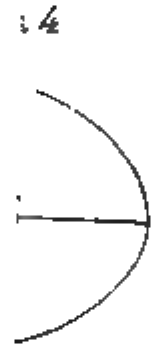

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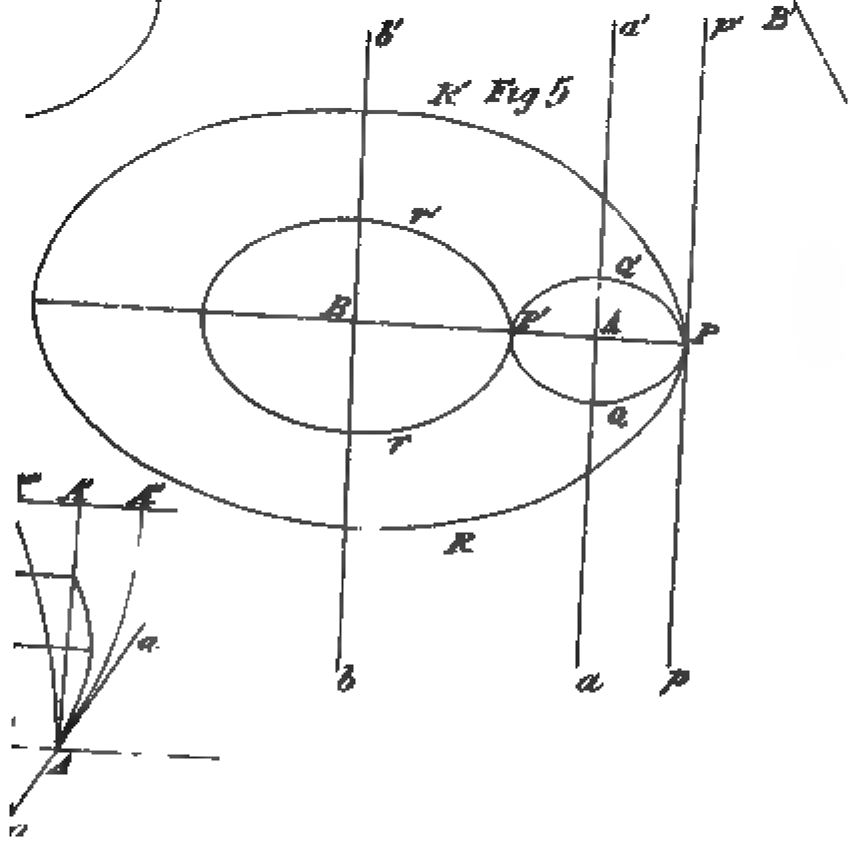


Fig 5

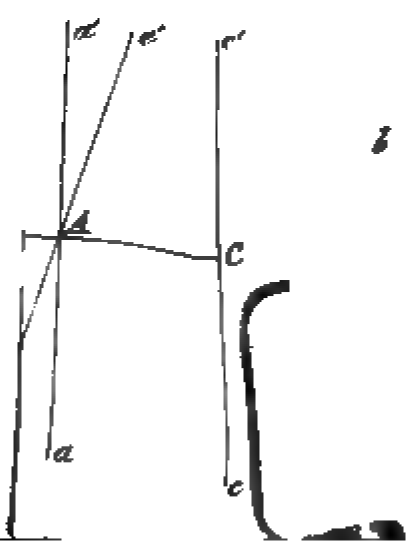


Fig. 8.

Fig.9.

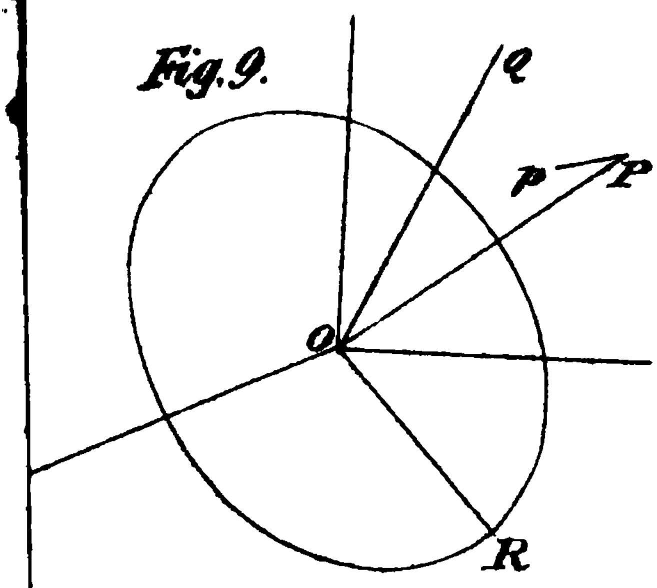
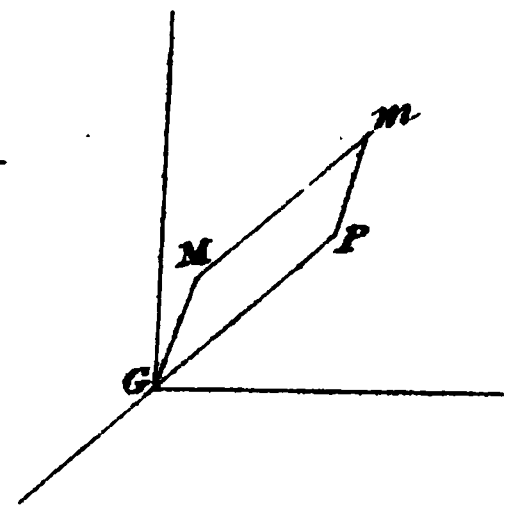


Fig 12



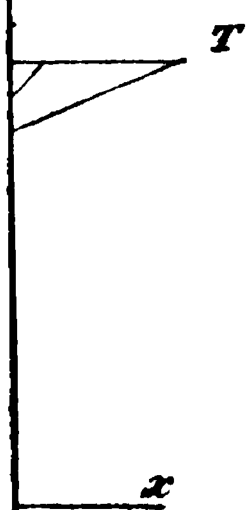
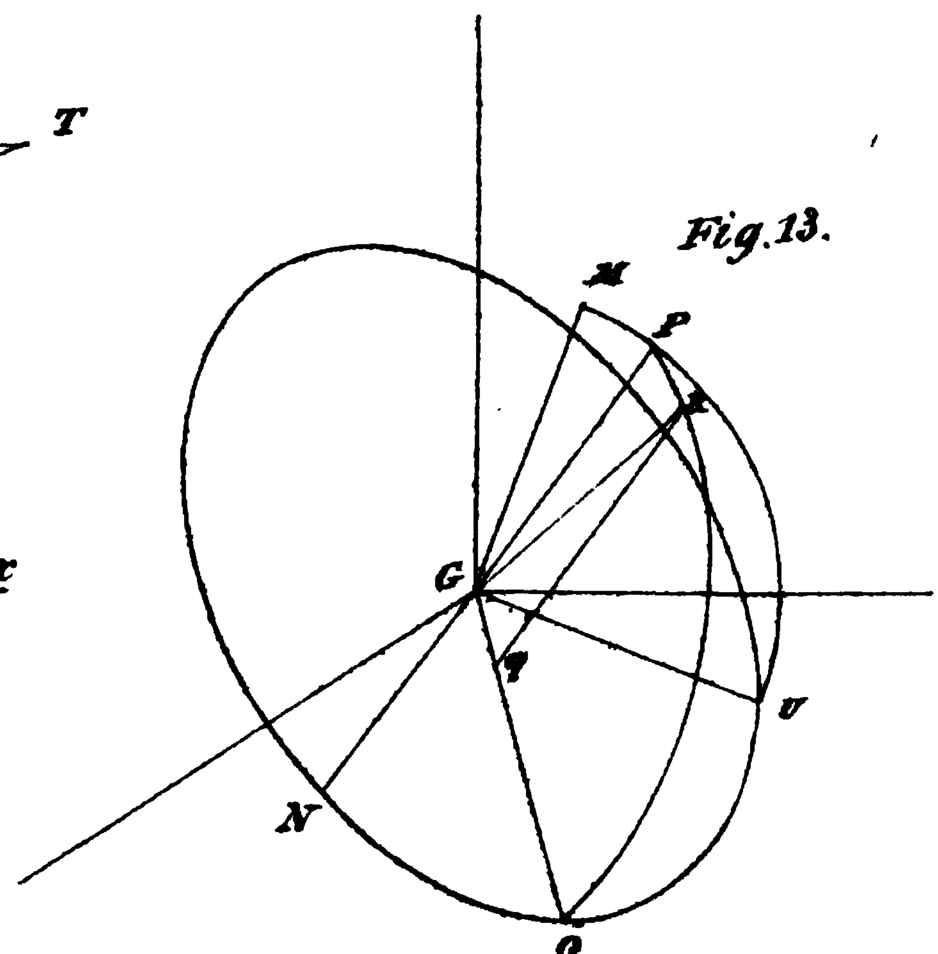


Fig.13.



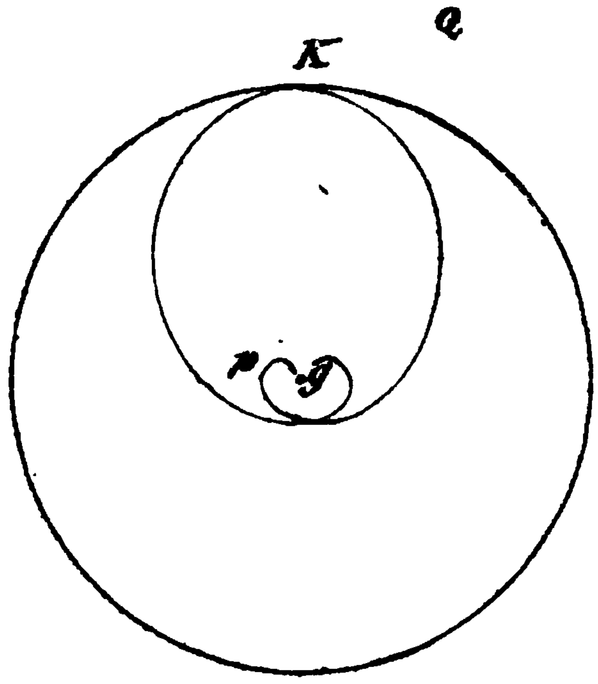
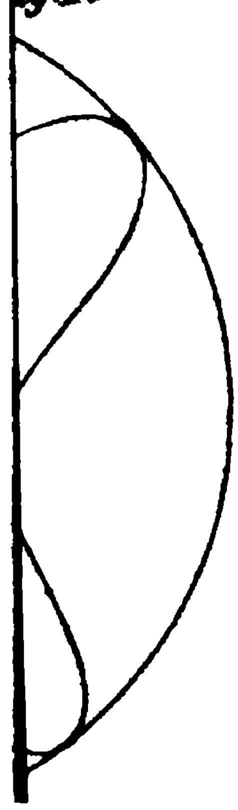


Fig 16

Fig.15.



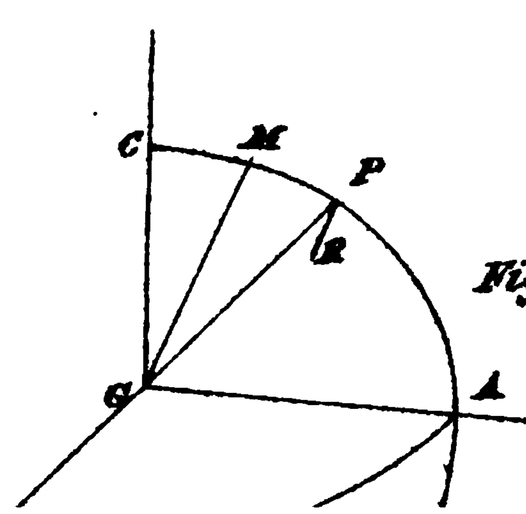
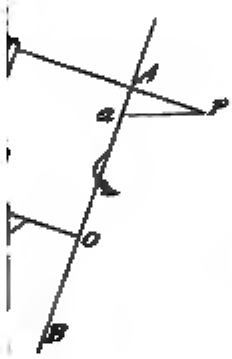


Fig.17.

Fig.19.



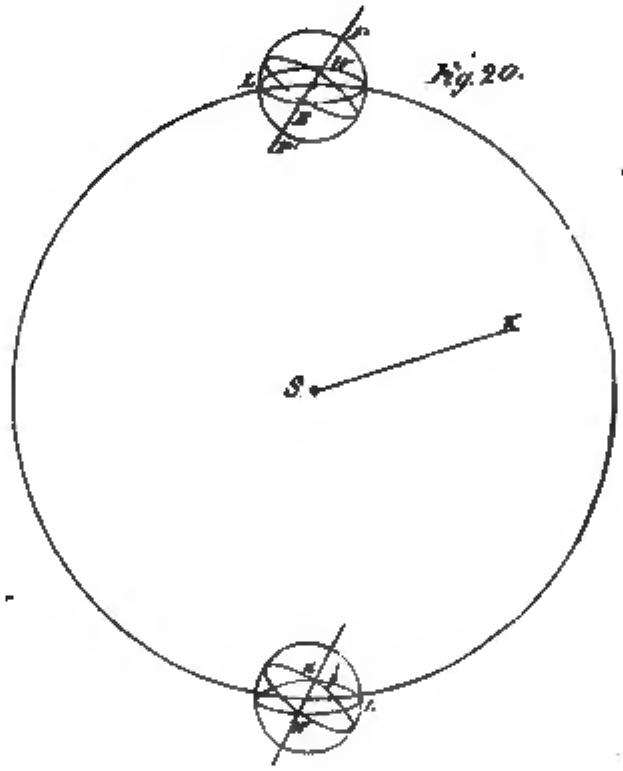
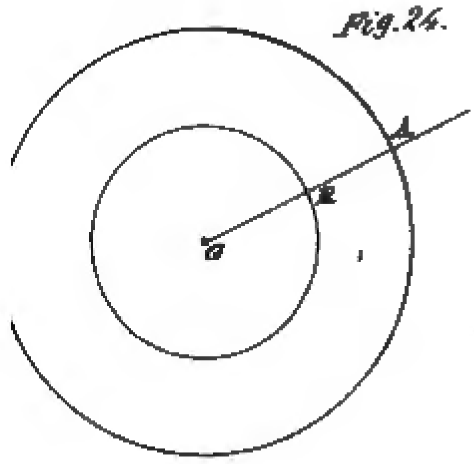
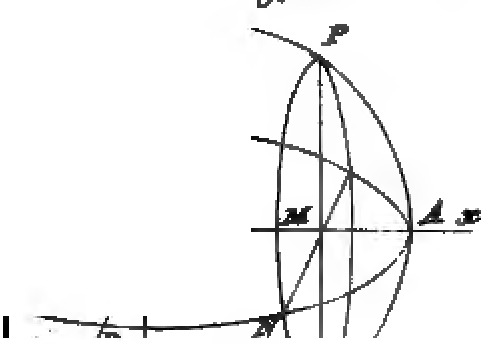


Fig.20.

Fig.24.



ig.25.



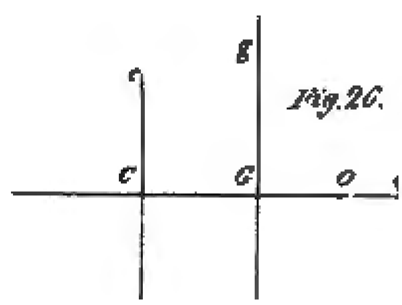


Fig.26.

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